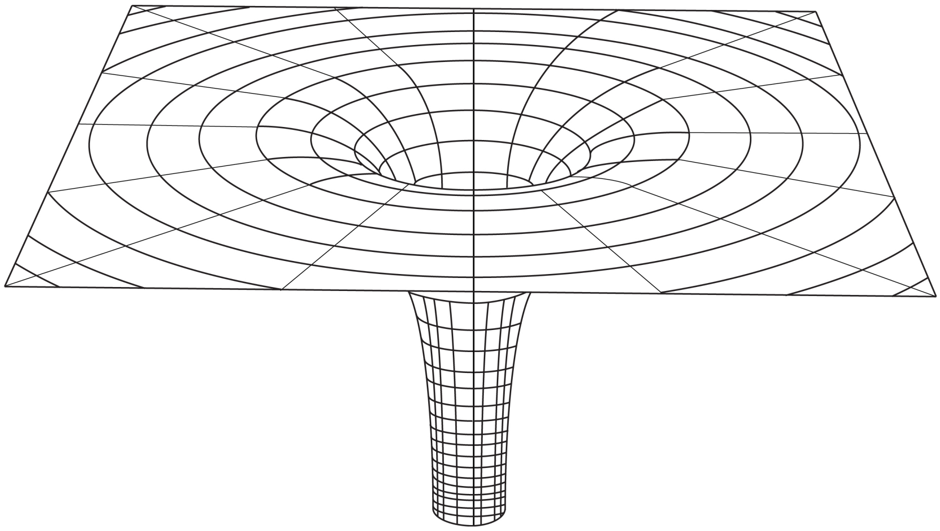

A cosmological study of *f(R,L)* theories

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Orientador

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Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

Porto, / /

"*In the beginning the Universe was created. This has made a lot of people very angry and been widely regarded as a bad move.*"

Douglas Adams, *The Restaurant at the End of the Universe*

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UNIVERSIDADE DO PORTO

Abstract

Departamento de Física e Astronomia
Faculdade de Ciências da Universidade do Porto

Master of Science

A cosmological study of $f(R, \mathcal{L})$ theories

by Rui Pedro Lopes de AZEVEDO

Recent cosmological evidence has provoked a resurgence of $f(R)$ and non-minimally coupled theories, amongst other extensions to General Relativity, as candidates to explain certain phenomena such as dark matter and dark energy. Following from these models, one can ponder the case of generic $f(R, \mathcal{L})$ theories, in which the Einstein-Hilbert action is replaced by an arbitrary function of the scalar curvature and the matter Lagrangian density.

In this work, we used a dynamical system approach to analyse the viability of $f(R, \mathcal{L})$ theories as candidates for dark energy. Dynamical system analysis is a very useful method for determining asymptotic solutions, as well as their stability, for systems of complex ordinary differential equations, such as the case of more elaborate cosmological models.

We studied the solutions for exponential and power-law models, and compared them against General Relativity and non-minimally coupled $f(R)$ models presented in previous works. An exponential model did not present any viable fixed points to explain dark energy. A power law, on the other hand, presented a couple of interesting points, one of which could be related with the recent accelerated expansion phase of the Universe.

Also present in this work is the proposal of a new model with Lagrangian density $f(R)\,(\kappa R + \mathcal{L})$, which presents several interesting characteristics and is worthy of further study. A preliminary dynamical analysis was performed on exponential and power-law functions for this model, and both presented solutions are capable of explaining accelerated expansion phases, both current and in the early universe.

The work relating to generic $f(R, \mathcal{L})$ theories (Chapters 2 and 3) was published in *Physical Review D* and can be found in Ref. [1], while the $f(R)\,(\kappa R + \mathcal{L})$ model (Chapter 4) is part of ongoing work and will be submitted in the near future.

UNIVERSIDADE DO PORTO

Resumo

Departamento de Física e Astronomia
Faculdade de Ciências da Universidade do Porto

Mestre de Ciência

Um estudo cosmológico de teorias $f(R, \mathcal{L})$

por Rui Pedro Lopes de AZEVEDO

Observações cosmológicas recentes provocaram uma resurgência de teorias $f(R)$ e com acoplamentos não-mínimos, entre outras extensões da Relatividade Geral, como candidatos para matéria e energia escuras. Partindo destes modelos, podemos considerar o caso de teorias $f(R, \mathcal{L})$ genéricas, nas quais a ação de Eisntein-Hilbert é substituida por uma função arbitrária da curvatura escalar e da densidade Lagrangeana da matéria.

Neste trabalho usámos uma abordagem de sistemas dinâmicos para analisar a viabilidade de teorias $f(R, \mathcal{L})$ como candidatos para energia escura. A análise de sistemas dinâmicos é um método muito eficiente para determinar soluções assimptóticas, assim como a estabilidade, de sistemas complexos de equações diferenciais ordinárias, tal como é o caso de modelos cosmológicos mais elaborados.

Estudámos soluções para modelos exponenciais e monomiais, e fizemos uma comparação com a Relatividade Geral e modelos $f(R)$ com acoplamento não-mínimo estudados em trabalhos anteriores. O modelo exponencial não apresentou nenhum ponto fixo viável para servir como candidato a energia escura. Por outro lado, o modelo monomial apresentou algums pontos interessantes, um dos quais pode estar associado à recente expansão acelerada do Universo.

Também presente neste trabalho é a proposta de um novo modelo com densidade Lagrangeana $f(R)\,(\kappa R + \mathcal{L})$, que apresenta várias características interessantes e que se mostra merecedor de estudos futuros. Uma análise dinâmica preliminar com funções exponenciais e monomias neste modelo revelou várias soluções capazes de explicar fases de expansão acelerada, tanto actual como no universo primitivo.

O trabalho referente a teorias genéricas $f(R, \mathcal{L})$ (Capítulos 2 e 3) foi publicado na revista *Physical Review D* e pode ser encontrado na Ref. [1], enquanto que o modelo $f(R)\,(\kappa R + \mathcal{L})$ (Capítulo 4) faz parte de trabalho em curso e será submetido em breve.

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GR	**G**eneral **R**elativity
EOS	**E**quation **O**f **S**tate
NMC	**N**on-**M**inimally **C**oupled
CDM	**C**old **D**ark **M**atter
FLRW	**F**riedmann-**L**emaître-**R**obertson-**W**alker
CMB	**C**osmic **M**icrowave **B**ackground
ODE	**O**rdinary **D**ifferential **E**quation
SNeIa	**S**uper**N**ova**e** type **Ia**
WMAP	**W**ilkinson **M**icrowave **A**nisotropy **P**robe
JBD	**J**ordan-**B**rans-**D**icke

Chapter 1

Introduction

Albert Einstein's General Relativity (GR) has served as the framework for the development of the so called standard model of cosmology. It is the simplest theory that relates matter and the curvature of spacetime, and by far the one with most experimental support [2, 3], from the prediction of the precession of Mercury's perihelion to the recent detection of gravitational wave production by black hole binaries [4].

Despite this backing, when coupled only with baryonic matter, GR still fails to account for more recent observations of the Universe. Comparisons of the rotational speed and mass of galaxies as predicted by GR and as measured via electromagnetic radiation do not appear to match, as if there was some missing mass from our calculations. Moreover, in the past two decades observations of supernovae have signalled that the Universe is expanding at an accelerating rate [5]. To address these flaws, the ΛCDM model was formulated, consisting of a universe evolving under GR and with the addition of dark energy, represented by a cosmological constant Λ with negative Equation of State (EOS) and that is responsible for this accelerated expansion, and Cold Dark Matter (CDM), a non-baryonic type of matter that either does not interact electromagnetically or has a vanishingly small interaction, which is responsible for this missing mass. This model is also supplemented by an inflationary scenario based on a scalar field to explain the early exponential expansion of the Universe.

An alternative to this solution is to assume that GR is incomplete, prompting other models to appear and attempt to explain this large scale behaviour. Among the most prominent are the so-called $f(R)$ theories [6–12], where the Einstein-Hilbert action is replaced by a nonlinear function of the scalar curvature, and models that present non-minimal couplings (NMC) between matter and curvature [13–17]. Some of these models have been shown to be able to mimic dark matter [18–21] or dark energy [22–24], and explain post-inflationary preheating [25] and cosmological structure formation [26–28].

Previous attempts at solving these cosmological problems using a NMC model have resorted to a coupling between curvature and a scalar field [29–37], but did not extend this coupling to the baryonic matter content. More recently, a dynamical system analysis approach was used to analyse a model that incorporated both $f(R)$ theories and a NMC with the baryonic matter content [38].

Taking this research background into account, in this thesis we do a dynamical system approach on a more general $f(R,\mathcal{L})$ group of theories [39], that allow for more non-linear couplings between matter and curvature. This method, on which we will elaborate further in the following sections, allows us to check for the existence of solutions to the cosmological equations, and to analyse their stability. Other similar studies, albeit in a different context, can be found in Refs. [40–43].

The organization of this work is as follows: in the following sections of Chapter 1 we provide an introduction to the method of dynamical analysis and to relevant gravitational models for later comparison and verification; in Chapter 2 we delve into a gravitational model of generic $f(R,\mathcal{L})$ theories, and perform the derivation of the dynamical system; Chapter 3 exposes the results of the dynamical analysis for two different $f(R,\mathcal{L})$ functions; Chapter 4 proposes a new NMC model and does a preliminary dynamical system analysis; finally, the overall conclusions of the thesis are presented in Chapter 5.

1.1 General Relativity

It is useful to review how the standard cosmological model is derived from GR before proceeding into the study of $f(R,\mathcal{L})$ theories. GR can be fully derived from the action

$$S=\int d^4x\sqrt{-g}\left[\kappa(R-2\Lambda)+\mathcal{L}\right], \tag{1.1}$$

where g is the determinant of the metric, R is the Ricci scalar, $\mathcal{L}$ is the matter Lagrangian density, Λ is the cosmological constant, and $\kappa=c^4/(16\pi G)$, with c the velocity of light in vacuum and G the Newton's gravitational constant.

We can derive the field equations by imposing a null variation of the action, $\delta S=0$, with respect to the metric $g_{\mu\nu}$, yielding the well-known Einstein field equations,

$$G_{\mu\nu}+\Lambda g_{\mu\nu}=\frac{1}{2\kappa}T_{\mu\nu}, \tag{1.2}$$

where $G_{\mu\nu}\equiv R_{\mu\nu}-g_{\mu\nu}R/2$ is the Einstein tensor and $T_{\mu\nu}$ is the matter energy-momentum tensor, defined as

$$T_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}\right)}{\delta g^{\mu\nu}}. \tag{1.3}$$

We can take the covariant derivative of the field equations and the Bianchi identities to obtain the conservation law for the energy-momentum tensor

$$\nabla^{\mu}T_{\mu\nu}=0. \tag{1.4}$$

Considering the Cosmological Principle, *i.e.* that the Universe is homogeneous and isotropic, and that the Universe is also flat, it can be well described via a

Friedmann-Lemaître-Robertson-Walker (FLRW) metric, represented by the line element

$$ds^2 = -dt^2 + a^2(t)dV^2, \tag{1.5}$$

where $a(t)$ is the scale factor and dV is the volume element in comoving coordinates, and where we set $c = \hbar = 1$ for the remainder of the work. From a cosmological standpoint, the matter content of the Universe can be described as a perfect fluid, with energy-momentum tensor

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{1.6}$$

derived from the Lagrangian density $\mathcal{L} = -\rho$ (see Refs. [44–46] for a discussion), where ρ and p are, respectively, the energy density and pressure of the perfect fluid, and u^μ is its four-velocity, with the normalization condition $u_\mu u^\mu = -1$. The pressure and energy density are considered to obey a equation of state (EOS) $p = w\rho$, where w is the EOS parameter.

By substituting this Lagrangian density and energy-momentum tensor in the conservation equation (1.4), we obtain the continuity equation

$$\dot{\rho} + 3H(1 + w)\rho = 0, \tag{1.7}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. By direct integration one obtains the general solution for ρ

$$\rho(t) = \rho_0 a(t)^{-3(1+w)}, \tag{1.8}$$

where ρ_0 is the value of the energy density at time t_0 and $a(t_0) \equiv 1$.

Introducing the metric (1.5) into field equations (1.2) we obtain the Friedmann and Raychaudhuri equations, respectively

$$H^2 = \frac{\rho}{6\kappa} + \frac{\Lambda}{3}, \tag{1.9}$$

$$2\dot{H} + 3H^2 = \Lambda - \frac{w\rho}{2\kappa}, \tag{1.10}$$

and the Ricci scalar yields

$$R = 6\left(2H^2 + \dot{H}\right). \tag{1.11}$$

Our current understanding of the Universe leads us to postulate the existence of four stages in its evolution. Soon after the Big Bang, the Universe entered an inflationary period of exponential expansion, necessary to solve the horizon, flatness and monopole problems, and to provide a mechanism that explains the homogeneity we observe in the Cosmic Microwave Background (CMB) and large structure formation [47]. Describing this period requires the addition of a scalar field, but is nonetheless well described by GR.

After inflation the Universe became dominated by radiation, in which the CMB was formed and can be described by taking $\rho \simeq \rho_R \gg \Lambda$ and $w_R = p_R/\rho_R = 1/3$. Via the continuity (1.7) and Friedmann (1.9) equations we obtain $a(t) \propto t^{1/2}$ and $\rho_R \propto a^{-4} \propto t^{-2}$ [48]. Following recombination came an era of matter domination. Similarly to radiation, it is described by $\rho \simeq \rho_M \gg \Lambda$, but with $w_M = p_M/\rho_M = 0$, which leads to the solutions $a(t) \propto t^{2/3}$ and $\rho_M \propto a^{-3} \propto t^{-2}$.

Following the transition period we are currently experiencing, the Universe will most likely be dominated by dark energy, in which case $\Lambda \gg \rho$ and the Hubble parameter is constant, leading to an exponential expansion $a(t) \propto \exp(H_0 t)$.

1.1.1 Brief Review of Dynamical System Analysis

Even though we can sometimes arrive directly at the solutions for the scale factor from the field equations, these solutions are often limited to special conditions, and only sometimes provide clues about their stability. However, using a dynamical system approach we can derive both the solutions and the conditions that they satisfy, as well as analyse their stability. A great review on the use of dynamical systems in cosmology can be found in Refs. [49, 50]. We present here the most relevant definitions used in the analyses that are performed throughout this thesis.

Definition 1 A *singular* or *fixed point* of a system of autonomous ordinary differential equations (ODEs)

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{1.12}$$

is a point $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ such that $\boldsymbol{f}(\bar{\boldsymbol{x}}) = 0$.

Definition 2 A fixed point $\bar{x}$ is called a *hyperbolic* fixed point if $Re(\lambda_i) \neq 0$ for all eigenvalues λ_i of the Jacobian of the vector field $\boldsymbol{f}(\boldsymbol{x})$ evaluated at $\bar{\boldsymbol{x}}$. Otherwise the point is called *non-hyperbolic*.

In order to analyse a given system, one must first locate its fixed points. Once that task is completed, one can then consider the behaviour of the system is the neighbourhood of each of the points. Assuming that the vector field $\boldsymbol{f}(\boldsymbol{x})$ is of class C^1, the process of determining its local behaviour consists of a linear approximation of the field in the neighbourhood of the fixed point $\bar{\boldsymbol{x}}$. In a close enough vicinity of this point,

$$\boldsymbol{f}(\boldsymbol{x}) \approx D_f(\bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}}), \tag{1.13}$$

where $D_f(\bar{\boldsymbol{x}})$ is the Jacobian matrix of the vector field evaluated at the fixed point $\bar{\boldsymbol{x}}$. Each of the fixed points can then be classified according to the eigenvalues of the Jacobian at that fixed point, according to the Hartman-Grobman Theorem [51]:

Hartman-Grobman Theorem Consider a system of ODEs $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, where the vector field $\boldsymbol{f}$ is of class C^1. If $\bar{\boldsymbol{x}}$ is a hyperbolic fixed point of the ODEs then

TABLE 1.1: Fixed points and respective solutions for GR.

$Point$	$(\Omega_M, \Omega_\Lambda)$	$a(t)$	$\rho(t)$	q
$\mathcal{A}$	$(1,0)$	$\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}$	$\rho_0\left(\frac{t}{t_0}\right)^{-2}$	$\frac{1+3w}{2}$
$\mathcal{B}$	$(0,1)$	$\exp\left(\sqrt{\Lambda/3}\,t\right)$	0	-1

there exists a neighbourhood of $\bar{\boldsymbol{x}}$ on which the flow is topologically equivalent to the flow of the linearisation of the ODEs at $\bar{\boldsymbol{x}}$.

If we consider a linear system of ODEs:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, \tag{1.14}$$

where A is a matrix with constant coefficients, it is immediate that if all the eigenvalues of the matrix A are positive, then the solutions in the neighbourhood of $\bar{\boldsymbol{x}} = 0$ will diverge from the fixed point $\bar{\boldsymbol{x}} = 0$. Conversely, if the same eigenvalues are all negative, the solutions will converge to the fixed point.

The Hartman-Grobman Theorem allows us to expand this classification to non-linear systems via their linearisation, and it is this method we use throughout this work to classify the stability of solutions to the field equations to small perturbations. If the eigenvalues of the Jacobian are all negative, the point is stable; if they are all negative, the fixed point is unstable; if there are both positive and negative eigenvalues, it corresponds to a saddle point.

1.1.2 Dynamical System Formulation of GR

As a first step, one must rewrite the Friedmann equation (1.9) in a dimensionless form

$$1 = \frac{\rho}{6\kappa H^2} + \frac{\Lambda}{3H^2} = \Omega_M + \Omega_\Lambda, \tag{1.15}$$

where $\Omega_M = \rho/(6\kappa H^2)$ and $\Omega_\Lambda = \Lambda/(3H^2)$ are the relative matter and dark energy densities, respectively. Next, we differentiate these variables with respect to the number of e-folds $N = \ln a(t)$, and obtain the autonomous system

$$\begin{cases} \frac{d\Omega_M}{dN} = 3\Omega_M\left[w(\Omega_M - 1) - \Omega_\Lambda\right] \\ \frac{d\Omega_\Lambda}{dN} = 3\Omega_\Lambda(1 + w\Omega_M - \Omega_\Lambda) \end{cases},$$

on which the Friedmann equation (1.15) acts as an algebraic constraint. We can obtain the fixed points of the system by imposing a null variation of the variables. The points and corresponding solutions can be found in Table 1.1.

In this case, one finds that the matter dominated point $\mathcal{A}$ is unstable, while the dark energy dominated one point $\mathcal{B}$ is stable. This is consistent with our understanding of the present day Universe, as we are going from an unstable matter dominated era to an apparently stable dark energy dominated one. Note that point $\mathcal{A}$ includes solutions for both matter ($w = 0$) and radiation ($w = 1/3$).

The scale factor, and subsequently the density, were determined using the variable definitions (1.15) and the relation between the Ricci scalar and the Hubble parameter Eq. (1.11). The deceleration parameter was obtained from its definition

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2}. \tag{1.16}$$

1.1.3 Inflation

When studying GR in a cosmological scale, it is also important to discuss the inflationary epoch in more detail. Historically, inflation was developed as a solution for the following problems [47]:

Flatness problem

The Friedmann equation for a curved universe can be generally written in a form similar to Eq. (1.15)

$$\Omega - 1 = \frac{K}{a^2 H^2}, \tag{1.17}$$

where Ω is the sum of all the relative matter densities and K is the curvature of the universe. If the Universe is flat, $K = 0 \to \Omega = 1$, and it remains so for all time. However, since during matter- or radiation-dominated phases we have $|\Omega - 1| \propto t^{\alpha}$, and observational data places the present day value of Ω extremely close to 1, we would require it to be even closer to 1 at very early times. For example, at nucleosynthesis, we would need $|\Omega(t_{nuc}) - 1| \lessapprox 10^{-16}$. As such, the flatness problem boils down to a fine tuning problem, as the necessary conditions for the evolution of the Universe towards its present state would have been extremely unlikely.

Horizon problem

This problem is closely related to CMB observations, and particularly to the homogeneity of radiation temperature all over the night sky. Since in the Hot Big Bang model the distance over which causal interaction can occur before decoupling is smaller than the distance that radiation travels after being emitted, one would expect that microwaves coming from causally separated regions to have different temperatures. A similar problem occurs in nucleosynthesis.

Homogeneity and isotropy

While we consider the Universe to be homogeneous at very large scales, it is not perfectly so, or there would have been no large-scale structure formation. The most plausible explanation for the small inhomogeneities in the CMB are irregularities at the surface of last scattering, that are impossible to generate causally via the Hot Big Bang model.

Unwanted relics

The last motivator for inflation consists in the possibility of particles or topological defects generated at very high temperatures surviving to the present, contrary to their current observational constraints.

The simplest solution to all these problems is to add an inflationary epoch just after the big bang, in which the scale factor increases by a large amount in a short time, usually given by the number of e-folds N, defined as

$$N(t) \equiv \ln \frac{a(t_{end})}{a(t)}, \tag{1.18}$$

where t_{end} is the time at the end of inflation. This number represents the amount of inflation that still has to occur after time t, with N decreasing to 0 as t approaches t_{end}. Resolution of the aforementioned problems requires about 70 e-folds before inflation ends.

This period is usually described via a homogeneous scalar field $\phi(t)$ dubbed the "inflaton", with energy density and pressure given by

$$\rho_\phi = \tfrac{1}{2}\dot{\phi}^2 + V(\phi), \tag{1.19}$$

$$p_\phi = \tfrac{1}{2}\dot{\phi}^2 - V(\phi), \tag{1.20}$$

where $V(\phi)$ is the potential of the scalar field.

Substituting in the Friedmann (1.9) and continuity (1.7) for an inflaton filled universe we obtain

$$H^2 = \frac{1}{6\kappa}\left[V(\phi) + \tfrac{1}{2}\dot{\phi}^2\right], \tag{1.21}$$

and

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \tag{1.22}$$

where primes represent differentiation with respect to the scalar field. For inflation to occur, one must have $\dot{\phi}^2 < V(\phi)$.

Slow-Roll Approximation

It is in this context that the slow-roll approximation was developed, where one writes

$$H^2 \simeq \frac{1}{6\kappa}V(\phi), \tag{1.23}$$

and

$$3H\dot{\phi} \simeq -V'(\phi). \tag{1.24}$$

These approximations hold if one requires that the relative change in the Hubble parameter H and in the field variation $\dot{\phi}$ be very small over an expansion time $1/H$, *i.e.*

$$|\dot{H}| \ll H^2, \tag{1.25}$$

$$|\ddot{\phi}| \ll H|\dot{\phi}|. \tag{1.26}$$

Immediately the later leads to Eq. (1.24), while the former returns $\dot{\phi}^2 \ll V(\phi)$, allowing the simplification of Eq. (1.21) into Eq. (1.23). One can then write these conditions as

$$\frac{|\dot{H}|}{H^2} = \epsilon(\phi) = \kappa \left(\frac{V'}{V}\right)^2 \ll 1, \tag{1.27}$$

$$\frac{|\ddot{\phi}|}{H|\dot{\phi}|} = \eta(\phi) = 2\kappa \frac{V''}{V} \ll 1, \tag{1.28}$$

where ϵ and η are called the slow-roll parameters.

In this regime, the deceleration parameter (1.16) becomes

$$q = -1 - \frac{\dot{H}}{H^2} = -1 + \kappa \left(\frac{V'}{V}\right)^2 = -1 + \epsilon, \tag{1.29}$$

and as per Eq. (1.27) is negative, and thus inflation is always guaranteed.

A proper inflationary model then consists of a defined field potential and a way of ending inflation, usually by breaking the slow-roll conditions. The previously described slow-roll model, proposed by Linde [52], constitutes what has become known as *new inflation*, to contrast it with Guth's original model [53], which has taken the name *old inflation*. Guth's model was far simpler, requiring the Universe to be trapped in a false vacuum that drove inflation and then tunnelling to the real vacuum in bubbles, which would in turn percolate and reheat the Universe. Another model later proposed by Linde is that of *hybrid inflation* [54], which resorts to two fields in order to drive and then resolve the inflationary period.

Reheating

During inflation, the massive increase in size of the Universe drastically lowers its temperature, so when it is over the Universe goes through a process called reheating, which finally gives way to the standard Big Bang evolution. As with inflation, this process is necessary for baryogenesis and the creation of density perturbations, though its specifics will depend on the dynamics of the inflationary field.

One can usually describe reheating through three phases:

Scalar field oscillations When the slow-roll conditions are violated at the end of inflation, the scalar field begins to oscillate around the minimum of the field potential. At this point, the potential can be approximated by a parabola and the oscillations can be described as a simple harmonic oscillator

$$\dot{\bar{\rho}}_\phi + 3H\bar{\rho}_\phi = 0, \tag{1.30}$$

where $\bar{\rho}_\phi = \langle \dot{\phi}^2 \rangle_t$ is the average field energy density. Since this is the same equation that describes the density of nonrelativistic matter, it is easy to see that the inflaton energy density decreases with a^{-3}.

Inflaton decay Next one has to account for the creation of the particles we now observe in the Universe. This is done by simply adding a decay term Γ_ϕ into the oscillation equation (1.30), such that

$$\dot{\bar{\rho}}_\phi + (3H + \Gamma_\phi)\bar{\rho}_\phi = 0, \tag{1.31}$$

and the energy of the inflaton flows into the ordinary matter radiation fields.

Thermalisation Finally, the decay products (which may include exotic particles) will interact, decay and then reach thermal equilibrium, and the Universe will then evolve as predicted by the Hot Big Bang model.

1.2 $f(R)$ Theories

From a phenomenological standpoint (and without the addition of more fields), the most logical way to generalise the Einstein-Hilbert action (1.1) is to replace the Ricci scalar R with a generic function $f(R)$

$$S = \int d^4x\sqrt{-g}\left[\kappa f(R) + \mathcal{L}\right]. \tag{1.32}$$

It is immediate that when $f(R) = (R - 2\Lambda)$, GR is recovered. We obtain the field equations of this action (1.32) using the same method as in GR, so that

$$G_{\mu\nu} f' = \frac{1}{2} g_{\mu\nu}[f - Rf'] + \Delta\mu\nu f' + \frac{1}{2\kappa} T_{\mu\nu}, \tag{1.33}$$

where primes denotes differentiation with respect to the Ricci scalar, $G_{\mu\nu}$ is the Einstein tensor and $\Delta_{\mu\nu} \equiv \nabla_\mu \nabla_\nu - g_{\mu\nu}\Box$, with $\Box = \nabla_\mu \nabla^\mu$ the D'Alembertian operator.

Since the Ricci scalar involves first and second order derivatives of the metric, the presence of $\Delta_{\mu\nu} f'$ in the field equations (1.33) makes them fourth order differential equations. If the action is linear in R, the fourth order terms vanish and the theory reduces to GR. There is also a differential relation between R and $T \equiv g^{\mu\nu} T_{\mu\nu}$, given by the trace equation

$$3\Box f' - 2f + Rf' = \frac{1}{2\kappa} T, \tag{1.34}$$

rather than the algebraic relation found in GR when $\Lambda = 0$, $R = -T/(2\kappa)$. This enables the admittance of a larger pool of solutions than GR, such as solutions that have scalar curvature, $R \neq 0$, when $T = 0$. The maximally symmetric solutions lead to a constant Ricci scalar, so for R constant and $T_{\mu\nu} = 0$, one obtains

$$Rf' - 2f = 0, \tag{1.35}$$

which is an algebraic equation in R for a given f. here it becomes important to distinguish between singular (R^{-n} , $n > 0$) and non-singular (R^n , $n > 0$) $f(R)$ models [55].

For non-singular models, $R = 0$ is always a possible solution, the field equations (1.33) reduce to $R_{\mu\nu} = 0$, and the maximally symmetric solution is Minkowski spacetime. When $R = C$ with C a constant, this becomes equivalent to a cosmological constant, the field equations reduce to $R_{\mu\nu} = g_{\mu\nu}C/4$, and the maximally symmetric solution is a de Sitter or anti-de Sitter space, depending on the sign of C. For singular $f(R)$ theories, however, $R = 0$ is no longer an admissible solution to Eq. (1.35), and one has to limit the impact of the extra Yukawa terms in the weak field expansion around R_0, which leads to the extra condition $f''(R_0) = 0$.

Similarly to GR, applying the Bianchi identities on the covariant derivative of the field equations yields the same conservation law for the energy-momentum tensor as in GR (1.4), $\nabla^\mu T_{\mu\nu} = 0$.

It is also possible to write the field equations (1.33) in the form of the Einstein equations with an effective stress-energy tensor

$$G_{\mu\nu} = \frac{1}{2\kappa f'}\left[T_{\mu\nu} + 2\kappa\nabla_{\mu\nu}f' + \kappa g_{\mu\nu}\left(f - Rf'\right)\right] \equiv \frac{1}{2\kappa f'}\left[T_{\mu\nu} + T^{(eff)}_{\mu\nu}\right] \tag{1.36}$$

where we can consider $G_{eff} \equiv G/f'$ to be the effective gravitational coupling strength, so that demanding that G_{eff} be positive returns $f' > 0$.

1.2.1 $f(R)$ Cosmology

As with any gravitational theory, in order for a $f(R)$ theory to be a suitable candidate for gravity, it must be compatible with the current cosmological evidence, and explain the observable cosmological dynamics, and generate cosmological perturbations compatible with microwave background, large scale structure and big bang nucleosynthesis.

To derive the modified Friedmann and Raychaudhuri equations we apply the same process as in Section 1.1.2. We assume a flat, homogeneous and isotropic universe describe by the FLRW metric in Eq. (1.5) and filled with a perfect fluid with energy density ρ, EOS $w = p/\rho$ and energy-momentum tensor defined in Eq. (1.6).

Inserting the metric and energy-momentum into the field equations (1.33), one obtains

$$H^2 = \frac{1}{3f'}\left[\frac{1}{2\kappa}\rho + \frac{Rf' - f}{2} - 3H\dot{f}'\right], \tag{1.37}$$

$$2\dot{H} + 3H^2 = -\frac{1}{f'}\left[\frac{1}{2\kappa}w\rho + \ddot{f}' + 2H\dot{f}' + \frac{f - Rf'}{2}\right]. \tag{1.38}$$

As seen earlier, we must have $f' > 0$ in order to have a positive G_{eff}. Adding to this condition, we need to have $f'' > 0$ to avoid ghosts [56] and the Dolgov-Kawasaki instability [57].

The usefulness of $f(R)$ theories becomes apparent if we introduce an effective density and pressure, defined as

$$\rho_{eff} = \kappa \left(\frac{Rf' - f}{f'} - \frac{6H\dot{f}'}{f'} \right), \tag{1.39}$$

$$p_{eff} = \frac{\kappa}{f'} \left(2\ddot{f}' + 4H\dot{f}' + f - Rf' \right), \tag{1.40}$$

where ρ_{eff} must be non-negative in a spatially flat FLRW spacetime for the Friedmann equation to have a real solution when $\rho \to 0$. When $\rho_{eff} \gg \rho$, the Friedmann (1.37) and Raychaudhuri (1.38) equations take the form

$$H^2 = \frac{1}{6\kappa}\rho_{eff}, \tag{1.41}$$

$$\frac{\ddot{a}}{a} = -\frac{q}{12\kappa}\left(\rho_{eff} + 3p_{eff}\right). \tag{1.42}$$

If we now consider the case of $f(R) \propto R^n$ and $a(t) = a_0(t/t_0)^\alpha$, the effective EOS parameter $w_{eff} = p_{eff}/\rho_{eff}$ and α become (for $n \neq 1$) [58]

$$w_{eff} = -\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3}, \tag{1.43}$$

$$\alpha = \frac{-2n^2 + 3n - 1}{n - 2}. \tag{1.44}$$

Now one can simply choose a value for n such that $\alpha > 1$ to obtain an accelerated expansion. One can also use data from supernovae and CMB observations to constrain the value of n [59]. Using the aforementioned model, type Ia supernovae (SNeIa) age tests give $-0.67 \leq n \leq -0.37$ or $1.37 \leq n \leq 1.43$, and data from the Wilkinson Microwave Anisotropy Probe (WMAP) constrain the same value at $-0.450 \leq n \leq -0.370$ or $1.366 \leq n \leq 1.376$, as the only ranges of values that fit the observational data while still having $\alpha > 1$ and a negative deceleration parameter.

More recently, the Planck collaboration has also set limits on this type of $f(R)$ theory [60], specifically in regards to the boundary condition B_0, which is the present day value of

$$B(z) = \frac{f''}{f'}\frac{H\dot{R}}{\dot{H} - H^2} = \frac{2(n-1)(n-2)}{-2n^2 + 4n - 3}. \tag{1.45}$$

Planck data implies that we must have $B_0 \lesssim 7.8 \times 10^{-5}$, so that $n \approx 1$ or $n \approx 2$.

One can also use $f(R)$ theories to describe inflation, for example using the well-known Starobinsky model [6,61] given by

$$f(R) = R + \frac{R^2}{6M^2}, \tag{1.46}$$

where M is a mass scale and where the presence of the linear term in R is responsible for bringing inflation to an end. The field equations (1.37) and (1.38) return

$$\ddot{H} - \frac{\dot{H}^2}{2H} + \tfrac{1}{2}M^2 H = -3H\dot{H}, \tag{1.47}$$

$$\ddot{R} + 3H\dot{R} + M^2 R = 0. \tag{1.48}$$

During inflation, the first two terms of Eq. (1.47) are much smaller than the other, and one obtains a linear differential equation for H that can be integrated to give

$$\begin{aligned} H &\simeq H_i - \frac{M^2}{6}(t - t_i), \\ a &\simeq a_i \exp\left[H_i(t - t_i) - \frac{M^2}{12}(t - t_i)^2\right], \\ R &\simeq 12H^2 - M^2, \end{aligned} \tag{1.49}$$

where H_i and a_i are the Hubble parameter and the scale factor at the onset of inflation ($t = t_i$), respectively. The slow-roll conditions imply

$$\epsilon = -\frac{\dot{H}}{H^2} \simeq \frac{M^2}{6H^2} \ll 1. \tag{1.50}$$

At the end of inflation, $\epsilon \simeq 1$, which implies that $H_f \simeq M/\sqrt{6}$ and $R \simeq M^2$. After inflation one requires once more a process to return the universe to the Hot Big Bang model and generate the particles we know today. In $f(R)$ theories this can happen in two ways: via a gravitational coupling to the particle fields in the perturbation regime, similarly to reheating in GR, or via a parametric resonance prior to the perturbative regime in a process dubbed *preheating* [62]. While we won't delve into its details, it is worth noting that particle production via preheating can occur for a wider range of the parameter space, thus avoiding some of the fine-tuning problems associated with standard reheating.

1.2.2 Dynamical System Formulation of $f(R)$ Theories

It is again useful to perform a dynamical analysis on $f(R)$ theories. This treatment is performed extensively in Ref. [42], so only the main results are presented here.

As before, one defines dimensionless variables

$$x = -\frac{\dot{f}'}{f'H}, \quad y = \frac{R}{6H^2}, \quad z = -\frac{f}{6f'H^2}, \quad \Omega = \frac{\rho}{6\kappa f' H^2}, \tag{1.51}$$

such that the Friedmann equation (1.37) takes the form

$$1 = x + y + z + \Omega. \tag{1.52}$$

Since the Ricci scalar takes the same form as in GR (1.11), one obtains from the y variable

$$\dot{H} = (y - 2)H^2, \tag{1.53}$$

and for a constant y, one can directly solve this equation for the scale factor.

$$a(t) = \begin{cases} \left(\frac{t}{t_0}\right)^{\frac{1}{2-y}}, & y \neq 2 \\ e^{H_0 t}, & y = 2 \end{cases}. \tag{1.54}$$

The deceleration parameter becomes

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = 1 - y, \tag{1.55}$$

so that from Eq. (1.11) the scalar curvature may be written as

$$R = 6H^2(1 - q). \tag{1.56}$$

Differentiating the variables (1.51) with respect to the number of e-folds N, one obtains the autonomous system

$$\begin{cases} \frac{dx}{dN} = & -1 + x^2 - y - xy - 3z + 3w\Omega \\ \frac{dy}{dN} = & y\left[2(2-y) - \frac{x}{\alpha}\right] \\ \frac{dz}{dN} = & z(x - 2y + 4) + \frac{xy}{\alpha} \\ \frac{d\Omega}{dN} = & \Omega(x - 2y + 1 - 3w) \end{cases}, \tag{1.57}$$

equivalent to the Friedmann and Raychaudhuri equations, (1.37) and (1.38). Solutions for fixed points can then be found when the $f(R)$ function is specified.

1.2.3 Equivalence with Scalar Field Theories

Another interesting factor of $f(R)$ theories is that they are equivalently to Jordan-Brans-Dicke (JBD) theories and scalar field theories [6]. The first equivalence is drawn by rewriting the $f(R)$ action (1.32) as a function of an arbitrary filed χ,

$$S = \int d^4x\sqrt{-g}\left[\kappa\left(f(\chi) + f'(\chi)(R - \chi)\right) + \mathcal{L}\right], \tag{1.58}$$

where primes represent differentiation with respect to χ. The null variation of this action with respect to χ returns

$$f''(\chi)(R - \chi) = 0, \tag{1.59}$$

which implies $\chi = R$ provided that $f''(\chi) \neq 0$, and therefore the action (1.58) takes the same form as Eq. (1.32).

Defining a new field $\varphi \equiv f'(\chi)$, Eq. (1.58) can be expressed as

$$S = \int d^4x\sqrt{-g}\left[\kappa\varphi R - V(\varphi) + \mathcal{L}\right], \tag{1.60}$$

where $V(\varphi)$ is a field potential given by

$$V(\varphi) = \kappa\left[\chi(\varphi)\varphi - f\left(\chi(\varphi)\right)\right], \tag{1.61}$$

which has the same form as the JBD action

$$S = \int d^4x\sqrt{-g}\left[\tfrac{1}{2}\varphi R - \frac{\omega_{JBD}}{2\varphi}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi) + \mathcal{L}\right], \tag{1.62}$$

when the JBD parameter ω_{JBD} is null.

It is also possible to write the $f(R)$ action as a scalar field theory in the Einstein frame via a conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{1.63}$$

where Ω^2 is the conformal factor and the tilde represents quantities in the Einstein frame. The Ricci scalars in each of the frames R and $\tilde{R}$ have the relation

$$R = \Omega^2\left(\tilde{R} + 6\tilde{\Box}\ln\Omega - 6\tilde{g}^{\mu\nu}\partial_\mu(\ln\Omega)\partial_\nu(\ln\Omega)\right). \tag{1.64}$$

Substituting in the action (1.60) and using the relation $\sqrt{-g} = \Omega^{-4}\sqrt{-\tilde{g}}$ we can rewrite it as

$$\begin{aligned} S = \int d^4x\sqrt{-\tilde{g}}\Big[\kappa\Omega^{-2}\varphi\Big(\tilde{R} + 6\tilde{\Box}(\ln\Omega) - 6\tilde{g}^{\mu\nu}\partial_\mu(\ln\Omega)\partial_\nu(\ln\Omega)\Big) \\ - \Omega^{-4}\varphi^2 U + \Omega^{-4}\mathcal{L}(\Omega^{-2}\tilde{g}_{\mu\nu}, \Psi_M)\Big], \end{aligned} \tag{1.65}$$

where now the matter Lagrangian is a function of the transformed metric $\tilde{g}_{\mu\nu}$ and the matter fields Ψ_M, and U is a potential defined as

$$U = \kappa\frac{\chi(\varphi)\varphi - f\left(\chi(\varphi)\right)}{\varphi^2}. \tag{1.66}$$

Careful observation of the previous equation makes it clear that one obtains the action in the Einstein frame for the choice of transformation

$$\Omega^2 = \varphi, \tag{1.67}$$

where it is assumed that $\varphi > 0$. We now rescale the scalar field as

$$\phi \equiv \sqrt{3\kappa}\ln(\varphi). \tag{1.68}$$

Since the integral $\int d^4x\sqrt{-\tilde{g}}\tilde{\Box}(\ln\Omega)$ vanishes due to Gauss's theorem, we can finally write the action (1.65) in the Einstein frame

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\kappa\tilde{R} - \tfrac{1}{2}\partial^\alpha\phi\partial_\alpha\phi - U(\phi) + e^{-2\frac{\phi}{\sqrt{3\kappa}}}\mathcal{L}\left(e^{-\frac{\phi}{\sqrt{3\kappa}}}\tilde{g}_{\mu\nu}, \Psi_M\right)\right], \tag{1.69}$$

where $e^{-\frac{\phi}{\sqrt{3\kappa}}}\tilde{g}_{\mu\nu}$ is the physical metric.

These three representations are equivalent in the absence of usual matter [63], so in this case one can choose to work in the representation that is more convenient. Scalar field an JBD theories may be more familiar to particle physicists, whereas the geometric nature of $f(R)$ may appeal more to mathematicians and relativists.

1.3 Non-minimally Coupled Theories

Following from $f(R)$, one can generalise the model by adding a non-minimal coupling between curvature and matter through the action

$$S = \int d^4x\sqrt{-g}\left[\kappa f_1(R) + f_2(R)\mathcal{L}\right], \tag{1.70}$$

with $f_1(R)$ and $f_2(R)$ arbitrary functions of the Ricci scalar R. Once again this model has already been extensively studied in Ref. [38], so we will only present the principal results here as reference for latter comparison. One recovers GR by taking $f_1(R) = R - 2\Lambda$ and $f_2(R) = 1$. The field equations are obtained as usual by imposing a null variation of the action with respect to the metric,

$$FG_{\mu\nu} = \frac{1}{2}f_2T_{\mu\nu} + \Delta_{\mu\nu}F + \frac{1}{2}g_{\mu\nu}\kappa f_1 - \frac{1}{2}g_{\mu\nu}RF, \tag{1.71}$$

where $F = \kappa f_1' + f_2'\mathcal{L}$, and the primes once again denote differentiation with respect to the scalar curvature. The Bianchi identities imply the noncovariant conservation law [17]

$$\nabla^\mu T_{\mu\nu} = \frac{f_2'}{f_2}\left(g_{\mu\nu}\mathcal{L} - T_{\mu\nu}\right)\nabla^\mu R. \tag{1.72}$$

Note that, even though Eq. (1.72) implies that energy is not generally conserved, the used metric (1.5) and energy-momentum tensor (1.6) make the right side of the conservation equation vanish, and one obtains the usual continuity equation (1.7). Moreover, and as pointed out in Ref. [45], one can show that if the Lagrangian of a particular matter field transforms as $\mathcal{L} \to A\mathcal{L}$ under a conformal transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where A is some power of Ω, then even if energy is not conserved in this frame, one finds conditions that must be satisfied for energy to be conserved in some other conformal frame

$$T = g_{\mu\nu}T^{\mu\nu} = 2\mathcal{L} \quad , \quad T^{\mu\nu} \to \tilde{T}^{\mu\nu} = \Omega^{-4}T^{\mu\nu} \quad , \quad \Omega^2 = f_2(R). \tag{1.73}$$

One must be careful when applying this procedure, since the conservation of energy depends not only on the choice of conformal transformation, but also on how the Lagrangian for a specific type of matter changes under that transformation.

The modified Friedmann and Raychaudhuri equations become

$$H^2 = \frac{1}{3F}\left[\frac{1}{2}FR - 3HF'\dot{R} - \tfrac{1}{2}\kappa f_1 + \frac{1}{2}f_2\rho - 9H^2(1+w)f_2'\rho\right], \tag{1.74}$$

$$2\dot{H} + 3H^2 = \frac{1}{2F}\left[FR - \kappa f_1 - 2\ddot{F} - 4H\dot{F} - f_2 w\rho\right]. \tag{1.75}$$

with $F' \equiv \kappa f_1'' - f_2''\rho$.

It should be noted that the presence of both $f(R)$ and a NMC terms can produce very interesting dynamics, both in the late and early universes. For instance, as is the case in $f(R)$ theories, once can achieve preheating processes following inflation [64].

1.3.1 Dynamical Analysis of NMC Theories

Once more one chooses dimensionless variables

$$\begin{aligned} x = -\frac{F'\dot{R}}{FH} \quad &, \quad y = \frac{R}{6H^2} \quad , \quad z = -\frac{\kappa f_1}{6FH^2}, \\ \Omega_1 = \frac{f_2\rho}{6FH^2} \quad &, \quad \Omega_2 = -\frac{3(1+w)f_2'\rho}{F}, \end{aligned} \tag{1.76}$$

such that the modified Friedmann equation can be read

$$1 = x + y + z + \Omega_1 + \Omega_2, \tag{1.77}$$

and the modified Raychaudhuri equation becomes

$$\frac{dx}{dN} + \frac{d\Omega_2}{dN} = (x+\Omega_2)(x+\Omega_2-y) - y - 3z + 3w\Omega_1 - 1. \tag{1.78}$$

Note that the introduction of the NMC increases the number of variables of the problem when compared with $f(R)$ models.

Following from the conservation law (1.7), one can derive the variables (1.76) with respect to the number of e-folds and obtain the autonomous system

$$\begin{cases} \frac{dx}{dN} = x\left[x - y + \Omega_2\left(1+\frac{\alpha_2}{\alpha}\right)\right] - 1 - y - 3z + 3w\Omega_1 + \Omega_2\left[3(1+w) - y\right] \\ \frac{dy}{dN} = y\left[2(2-y) - \frac{x}{\alpha}\right] \\ \frac{dz}{dN} = z\left[x\left(1-\frac{\alpha_1}{\alpha}\right) + \Omega_2 + 2(2-y)\right] \\ \frac{d\Omega_1}{dN} = \frac{\Omega_2 xy}{3\alpha(1+w)} + \Omega_1\left(1 - 3w + x + \Omega_2 - 2y\right) \\ \frac{d\Omega_2}{dN} = \Omega_2\left[x\left(1-\frac{\alpha_2}{\alpha}\right) - 3(1+w) + \Omega_2\right] \end{cases}, \tag{1.79}$$

where we have used the dimensionless parameters

$$\alpha(R,\rho) = \frac{F'R}{F}, \quad \alpha_1(R) = \frac{f_1'R}{f_1}, \quad \alpha_2(R) = \frac{f_2''R}{f_2'}. \tag{1.80}$$

Since the Raychaudhuri equation can be calculated by differentiating the Friedmann equation, and it is also equivalent to the relation for dx/dN, the relation

$$\frac{dx}{dN} + \frac{dy}{dN} + \frac{dz}{dN} + \frac{d\Omega_1}{dN} + \frac{d\Omega_2}{dN} = 0 \tag{1.81}$$

must hold. Fortunately, instead of vanishing trivially, Eq. (1.81) yields

$$y\left[\frac{\Omega_2}{3(1+w)} - 1\right] = z\alpha_1. \tag{1.82}$$

Relation (1.82) and the Friedmann equation (1.77) act on the system (1.79) as constraints, and allow us to reduce its dimensionality. Eliminating Ω_1 and Ω_2, we are left with

$$\begin{cases} \frac{dx}{dN} = & x\left[x - y + 3(1+w)\left(1+\frac{z}{y}\alpha_1\right)\left(1+\frac{\alpha_2}{\alpha}\right) - 3w\right] + 2(2+3w)(2-y) \\ & -3(1+w)z(1+\alpha_1) + 9(1+w)\frac{z}{y}\alpha_1 \\ \frac{dy}{dN} = & y\left[2(2-y) - \frac{x}{\alpha}\right] \\ \frac{dz}{dN} = & z\left[x\left(1-\frac{\alpha_1}{\alpha}\right) + 3(1+w)\left(1+\frac{z}{y}\alpha_1\right) + 2(2-y)\right] \end{cases} . \tag{1.83}$$

Solving this system generally also requires writing R and ρ as functions of the variables, which can be done recurring to the definition of the variables themselves. Specifically, the scalar curvature can be calculated from the inversion of the relation

$$\frac{f_2'(R)R}{f_2(R)} = -\frac{\Omega_2 y}{3(1+w)\Omega_1} = -\frac{y + z\alpha_1(R)}{\Omega_1}, \tag{1.84}$$

and the energy density from

$$\rho(y,z,\Omega_1) = -\frac{\kappa f_1\left(R(y,z,\Omega_1)\right)}{f_2\left(R(y,z,\Omega_1)\right)}\frac{\Omega_1}{z}. \tag{1.85}$$

All that one has to do now is to specify the functions $f_1(R)$ and $f_2(R)$ and solve for the fixed points and their stability. The results for several different functions can be found in Ref. [38].

1.3.2 Equivalence with Scalar Field Theories

In a similar way as shown before for $f(R)$ theories, one can rewrite NMC theories with an action with two scalar fields, or do a conformal transformation into the

Einstein frame. For the former it is enough to write the action

$$S=\int d^4x\sqrt{-g}\left[\kappa f_1(\chi)+\varphi(R-\chi)+f_2(\chi)\mathcal{L}\right], \tag{1.86}$$

where variation with respect to φ and χ give, respectively,

$$\chi=R, \tag{1.87}$$

$$\varphi=\kappa f_1'(\chi)+f_2'(\chi)\mathcal{L}, \tag{1.88}$$

which implies that both fields are independent if $\mathcal{L}\neq 0$ and $f_2'(\chi)\neq 0$. We can rewrite the action (1.86) in the form of a JBD type theory with $\omega_{JBD}=0$

$$S=\int d^4x\sqrt{-g}\left[\varphi R-V(\chi,\varphi)+f_2(\chi)\mathcal{L}\right], \tag{1.89}$$

with a potential

$$V(\chi,\varphi)=\varphi\chi-\kappa f_1(\chi). \tag{1.90}$$

Alternatively, one can make a conformal transformation $g_{\mu\nu}\to\tilde{g}_{\mu\nu}=\Omega^2 g_{\mu\nu}$ so that the action (1.89) reads

$$\begin{aligned}S=\int d^4x\sqrt{-\tilde{g}}\Big[&\kappa\Omega^{-2}\varphi\Big(\tilde{R}+6\tilde{\Box}(\ln\Omega)-6\tilde{g}^{\mu\nu}\partial_\mu(\ln\Omega)\partial_\nu(\ln\Omega)\Big)\\&-\Omega^{-4}\varphi^2U+\Omega^{-4}f_2(\chi)\mathcal{L}(\Omega^{-2}\tilde{g}_{\mu\nu},\Psi_M)\Big],\end{aligned} \tag{1.91}$$

where the potential U is given by

$$U=\kappa\frac{\varphi\chi-\kappa f_1(\chi)}{\varphi^2}. \tag{1.92}$$

To express the action in the Einstein frame, the conformal factor must now obey

$$\Omega^2=\varphi, \tag{1.93}$$

where it is assumed that $f_1'(R)>0$. We now rescale the two scalar fields as

$$\phi\equiv\sqrt{3\kappa}\ln(\varphi), \tag{1.94}$$

$$\psi\equiv f_2(\chi). \tag{1.95}$$

Once again Gauss's theorem allows us to finally write the action (1.91) as

$$\begin{aligned}S=\int d^4x\sqrt{-\tilde{g}}\Big[&\kappa\tilde{R}-\tfrac{1}{2}\partial^\alpha\phi\partial_\alpha\phi-U(\phi,\psi)\\&+\psi e^{-2\frac{\phi}{\sqrt{3\kappa}}}\mathcal{L}\left(e^{-\frac{\phi}{\sqrt{3\kappa}}}\tilde{g}_{\mu\nu},\Psi_M\right)\Big],\end{aligned} \tag{1.96}$$

where $e^{-\frac{\phi}{\sqrt{3\kappa}}}\tilde{g}_{\mu\nu}$ is the physical metric.

Chapter 2

$f(R, \mathcal{L})$ Theories

In the following chapters we proceed to study the broader generalization of the Einstein-Hilbert action inspired by $f(R)$ and NMC theories [6], positing an arbitrary non-minimal coupling between the matter Lagrangian and curvature, embodied in the action,

$$S = \int d^4x\sqrt{-g}\, f\left(R, \mathcal{L}\right), \tag{2.1}$$

where $f(R, \mathcal{L})$ is an arbitrary function of the scalar curvature R, the matter Lagrangian density $\mathcal{L}$ and g is the determinant of the metric. Much in the same way as with the previous models presented in Chapter 1, one can recover the particular cases of GR, $f(R)$ and NMC theories by setting $f(R, \mathcal{L}) = \kappa(R - 2\Lambda) + \mathcal{L}$, $f(R, \mathcal{L}) = \kappa f(R) + \mathcal{L}$ and $f(R, \mathcal{L}) = \kappa f_1(R) + f_2(R)\mathcal{L}$, respectively, with $\kappa = c^4/(16\pi G)$.

The action principle once again leads us to the field equations

$$f^R G_{\mu\nu} = \frac{1}{2} g_{\mu\nu}\left(f - f^R R\right) + \Delta_{\mu\nu} f^R + \frac{1}{2} f^L\left(T_{\mu\nu} - g_{\mu\nu}\mathcal{L}\right), \tag{2.2}$$

where

$$f^{\overbrace{R\ldots R}^{n}\overbrace{L\ldots L}^{m}} \equiv \frac{\partial^{n+m} f(R, \mathcal{L})}{\partial R^n \partial \mathcal{L}^m}. \tag{2.3}$$

The conservation law for the energy-momentum tensor takes a similar form to the one in Eq. (1.72),

$$\nabla^\mu T_{\mu\nu} = \left(g_{\mu\nu}\mathcal{L} - T_{\mu\nu}\right)\left(\frac{f^{RL}}{f^L}\nabla^\mu R + \frac{f^{LL}}{f^L}\nabla^\mu \mathcal{L}\right), \tag{2.4}$$

acting as a more general form of the former and showing once again that energy is not generally covariantly conserved.

Although not extensively studied, some work has already been done on $f(R, \mathcal{L})$ theories, ranging from general studies on the model [39, 65–67]and the addition of scalar fields [68], to cosmic strings [69].

2.1 $f(R,\mathcal{L})$ Cosmology

Once again we considered a flat universe subject to the cosmological principle, described by the FLRW metric (1.5) and filled with a perfect fluid with an energy-momentum tensor described by Eq. (1.6), derived from the Lagrangian density $\mathcal{L} = -\rho$.

By substituting this Lagrangian density and energy-momentum tensor in the conservation equation (2.4), one sees that energy is conserved for this particular case, and we once again obtain the usual continuity equation

$$\dot{\rho} + 3H(1+w)\rho = 0. \tag{2.5}$$

Note that since the purpose of this work is to find alternatives for dark energy using modified gravity, we inherently exclude exotic matter, *i.e.* $w < 0$, from making part of the composition of the universe. In fact, along with the solutions presented in this study, one finds that negative values of w are often capable of producing stable fixed points, but since they are also capable of this in GR, it would defeat the purpose of this thesis to include them.

As usual, inserting a FLRW metric into the field equations returns the modified Friedmann equation the 00 component

$$H^2 = \frac{1}{3f^R}\left[\frac{1}{2}f^R R - 3Hf^{RR}\dot{R} - \frac{1}{2}f - 9H^2 f^{RL}(1+w)\rho\right], \tag{2.6}$$

and the modified Raychaudhuri equation for the ii components

$$2\dot{H} + 3H^2 = \frac{1}{2f^R}\left[f^R R - f - f^L(1+w)\rho - 2\ddot{f^R} - 4H\dot{f^R}\right]. \tag{2.7}$$

2.2 Dynamical System Formulation

Dynamical analysis of $f(R)$ and NMC theories in Chapter 1 make it clear that one should choose dimensionless variables such that the modified Friedmann equation (2.6) becomes

$$1 = x + y + z + \phi, \tag{2.8}$$

with the variables defined as

$$\begin{gathered} x = -\frac{f^{RR}\dot{R}}{f^R H} \quad , \quad y = \frac{R}{6H^2} \quad , \quad z = -\frac{f}{6f^R H^2}, \\ \phi = -\frac{3(1+w)f^{RL}\rho}{f^R} \quad , \quad \theta = \frac{(1+w)f^L\rho}{2f^R H^2}. \end{gathered} \tag{2.9}$$

The quantities $\dot{f^R}/\left(f^R H\right)$ and $\ddot{f^R}/\left(f^R H^2\right)$ are useful in the subsequent derivations, so one should write them as functions of the variables (2.9):

$$\frac{\dot{f^R}}{f^R H} = -(x+\phi), \tag{2.10}$$
$$\frac{\ddot{f^R}}{f^R H^2} = (x+\phi)(x+\phi+2-y) - \frac{dx}{dN} - \frac{d\phi}{dN},$$

Rewriting the modified Raychaudhuri Eq. (2.7) as a function of the dimensionless variables defined above, the following relation may be obtained

$$\frac{dx}{dN} + \frac{d\phi}{dN} = (x+\phi)(x+\phi-y) - y - 3z + \theta - 1. \tag{2.11}$$

We can write this relation more explicitly by differentiating ϕ and using the continuity equation to obtain

$$\frac{d\phi}{dN} = \phi\left[x\left(1-\frac{\beta_R}{\alpha_R}\right) - 3(1+w)\left(\beta_L+1\right) + \phi\right], \tag{2.12}$$

and subsequently

$$\begin{aligned}\frac{dx}{dN} =& x\left[x-y+\phi\left(1+\frac{\beta_R}{\alpha_R}\right)\right] - 1 - y - 3z + \theta \\ &+ \phi\left[3(1+w)\left(\beta_L\right) - y\right].\end{aligned} \tag{2.13}$$

Through differentiation, we obtained the following autonomous system

$$\begin{cases} \frac{dx}{dN} = & x\left[x-y+\phi\left(1+\frac{\beta_R}{\alpha_R}\right)\right] - 1 - y - 3z + \theta \\ & +\phi\left[3(1+w)\left(\beta_L+1\right) - y\right] \\ \frac{dy}{dN} = & y\left[2(2-y) - \frac{x}{\alpha_R}\right] \\ \frac{dz}{dN} = & z\left[x+\phi+2(2-y)\right] + \frac{xy}{\alpha_R} - \theta \\ \frac{d\phi}{dN} = & \phi\left[x\left(1-\frac{\beta_R}{\alpha_R}\right) - 3(1+w)\left(\beta_L+1\right) + \phi\right] \\ \frac{d\theta}{dN} = & \frac{xy\phi}{\alpha_R} + \theta\left[x - 2y + \phi + 1 - 3w - 3(1+w)\gamma_L\right] \end{cases} \tag{2.14}$$

where we have made use of the dimensionless parameters

$$\eta_R = \frac{f^R R}{f}, \qquad \gamma_R = \frac{f^{RL} R}{f^L}, \qquad \gamma_L = -\frac{f^{LL}\rho}{f^L},$$
$$\alpha_R = \frac{f^{RR} R}{f^R}, \qquad \beta_R = \frac{f^{RRL} R}{f^{RL}}, \qquad \beta_L = -\frac{f^{RLL}\rho}{f^{RL}}, \tag{2.15}$$

and on which the modified Friedmann equation (2.8) acts as a constraint that can be used to reduce the dimensionality of the system. In this case, we chose to eliminate

z, and the system simplifies to

$$\begin{cases} \frac{dx}{dN} = & x\left[x-y+\phi\left(1+\frac{\beta_R}{\alpha_R}\right)+3\right]-4+2y+\theta \\ & +\phi\left[3(1+w)\left(\beta_L+1\right)+3-y\right] \\ \frac{dy}{dN} = & y\left[2(2-y)-\frac{x}{\alpha_R}\right] \\ \frac{d\phi}{dN} = & \phi\left[x\left(1-\frac{\beta_R}{\alpha_R}\right)-3(1+w)\left(\beta_L+1\right)+\phi\right] \\ \frac{d\theta}{dN} = & \frac{xy\phi}{\alpha_R}+\theta\left[x-2y+\phi+1-3w-3(1+w)\gamma_L\right] \end{cases} . \tag{2.16}$$

As with the NMC case, solving this system usually requires writing expressions for the scalar curvature R and the energy density ρ. The specific expressions for these quantities are of course model dependant, but in general they can be obtained from the relations

$$\begin{aligned} yf &= -zf^R R = (x+y+\phi-1)f^R R, \\ \theta f &= -3(1+w)zf^L\rho = 3(1+w)(x+y+\phi-1)f^L\rho, \\ \theta f^R R &= 3(1+w)yf^L\rho, \\ y\phi f &= 3(1+w)zf^{RL}R\rho = 3(1+w)(1-x-y-\phi)f^{RL}R\rho, \\ \phi f^R &= -3(1+w)f^{RL}\rho, \\ y\phi f^L &= -\theta f^{RL}R. \end{aligned} \tag{2.17}$$

It is important to note that this system is not always solvable or unique. As such, the functions $R = R(x, y, \phi, \theta)$ and $\rho = \rho(x, y, \phi, \theta)$ can only be written explicitly in models in which at least two of the previous equalities are non-trivial, invertible and distinct. Failure to find specific forms for these functions may prevent one from arriving at some of the more relevant physical quantities, even if one is able to solve the autonomous system (2.16) for its fixed points and determine their stability.

These quantities maintain the same relations to the scale factor as in the $f(R)$ and NMC cases, for a constant y, and we can simply read them from the previous results for the scale factor (1.54), deceleration parameter (1.55), Ricci scalar (1.11) and (1.56), and energy density (1.8).

2.3 Comparison with Other Models

Having established the dynamical system that follows from a generic $f(R,\mathcal{L})$ model, it is a good test to check if it produces the same results if we restrict the type of functions that we can use. In this section we do just that, by reproducing some of the results obtained in Chapter 1.

2.3.1 General Relativity

GR is recovered when $f(R,\mathcal{L}) = \kappa(R-2\Lambda)+\mathcal{L}$, so that

$$\begin{aligned} f^R &= \kappa, & f^L &= 1, \\ \beta_L &= 0, & \gamma_L &= 0, \\ \frac{x}{\alpha_R} &= -\frac{\dot{R}}{HR}, & x\phi\frac{\beta_R}{\alpha_R} &= 0, \end{aligned} \tag{2.18}$$

and higher order derivatives of $f(R,\mathcal{L})$ vanish. Substituting the variable definition (2.9) returns $x=\phi=0$ and $\theta = 4-2y = 3(1+w)$. The former relation, when combined with the constraint (2.8), heavily simplifies the modified Raychaudhuri Eq. (2.11), and yields the additional constraint

$$0 = 1+y+3z-\theta \rightarrow y = 2-\frac{\theta}{2}, \tag{2.19}$$

Finally, θ can be obtained by direct integration of the remaining equation of the dynamical system,

$$\frac{d\theta}{dN} = \theta[\theta-3(1+w)], \tag{2.20}$$

which has the solution

$$\theta(a) = \frac{3(1+w)}{1+\frac{2\kappa\Lambda}{\rho_0}\left(\frac{a}{a_0}\right)^{3(1+w)}}, \tag{2.21}$$

where ρ_0 is the density when $a=a_0$.

When the second term in the denominator, or equivalently, the scale factor, is small enough, we obtain $\theta = 3(1+w)$, and subsequently the remaining values of the fixed point $\mathcal{A}$, depicted on Table 2.1 using the previously determined relations, which corresponds to a matter filled universe and is unstable.

If, on the other hand, the scale factor and therefore the denominator are much larger, we get $\theta \rightarrow 0$ and $y \rightarrow 2$, implying an exponential scale factor driving a de Sitter expansion phase, indicated by point $\mathcal{B}$. These are exactly the same results obtained for GR in Sec. 1.1. From the definition of the dimensionless variables (2.9),

$$H_0^2 = \frac{\Lambda}{3(y+z)} = \frac{\Lambda}{3}. \tag{2.22}$$

2.3.2 Non-minimal Coupling

The correct derivation of the dynamical system for general $f(R,\mathcal{L})$ theories should also encompass the case of NMC theories presented in Sec. 1.3, defined by the function $f(R,\mathcal{L}) = \kappa f_1(R) + f_2(R)\mathcal{L}$, that in itself also includes $f(R)$ theories

TABLE 2.1: Fixed points and respective solutions for GR, obtained via the $f(R,\mathcal{L})$ dynamical system.

	(x,y,z,ϕ,θ)	$a(t)$	$\rho(t)$	q
$\mathcal{A}$	$\left(0,\frac{1-3w}{2},\frac{1+3w}{2},0,3(1+w)\right)$	$\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}$	$\rho_0\left(\frac{t}{t_0}\right)^{-2}$	$\frac{1+3w}{2}$
$\mathcal{B}$	$(0,2,-1,0,0)$	$e^{H_0 t}$	0	-1

when $f_2(R)=1$. With this function, the relations (2.17) yield the additional constraint

$$y\left[\frac{\phi}{3(1+w)}-1\right]=\left[z-\frac{\theta}{3(1+w)}\right]\frac{f_1^R R}{f_1}, \tag{2.23}$$

or equivalently,

$$\hat{y}\left[\frac{\hat{\Omega}_2}{3(1+w)}-1\right]=\hat{z}\hat{\alpha}_1, \tag{2.24}$$

where $\hat{\alpha}_1 = f_1^R R/f_1$ and the variables $(\hat{x},\hat{y},\hat{z},\hat{\Omega}_1,\hat{\Omega}_2)$ are the ones defined in Eqs. (1.76) and (1.80) for the NMC case, and are related to those defined in (2.9) by

$$\begin{aligned}
\hat{x}=x \quad , \quad \hat{y}=y \quad &, \quad \hat{z}=z-\frac{\theta}{3(1+w)},\\
\hat{\Omega}_1=\frac{\theta}{3(1+w)} \quad &, \quad \hat{\Omega}_2=\phi.
\end{aligned} \tag{2.25}$$

It comes as no surprise that this is precisely the constraint previously obtainable from the NMC modified Raychaudhuri equation in Eq. (1.82). All that is left to do is obtain the system (1.79) from system (2.14) by substituting the variables with the hatted ones defined above

$$\begin{cases}
\frac{d\hat{x}}{dN} = & \hat{x}\left[\hat{x}-\hat{y}+\hat{\Omega}_2\left(1+\frac{\hat{\alpha}_2}{\hat{\alpha}}\right)\right]-1-\hat{y}-3\hat{z}\\
& +3w\hat{\Omega}_1+\hat{\Omega}_2\left[3(1+w)-\hat{y}\right]\\
\frac{d\hat{y}}{dN} = & \hat{y}\left[2(2-\hat{y})-\frac{\hat{x}}{\hat{\alpha}}\right]\\
\frac{d\hat{z}}{dN} = & \hat{z}\left[\hat{x}\left(1-\frac{\hat{\alpha}_1}{\hat{\alpha}}\right)+\hat{\Omega}_2+2(2-\hat{y})\right]\\
\frac{d\hat{\Omega}_1}{dN} = & \frac{\hat{\Omega}_2\hat{x}\hat{y}}{3\hat{\alpha}(1+w)}+\hat{\Omega}_1\left(1-3w+\hat{x}+\hat{\Omega}_2-2\hat{y}\right)\\
\frac{d\hat{\Omega}_2}{dN} = & \hat{\Omega}_2\left[x\left(1-\frac{\hat{\alpha}_2}{\hat{\alpha}}\right)-3(1+w)+\hat{\Omega}_2\right]
\end{cases}. \tag{2.26}$$

2.4 $f(R,\mathcal{L})$ in a de Sitter Universe

An exponential scale factor is one of the most suitable models to explain an accelerated expansion of the universe. As such, instead of experimenting with different $f(R,\mathcal{L})$ models to try and replicate such scenario, we can work backwards and impose a de Sitter universe, which from Eq. (1.54) implies $y=2$. As we start for the desired fixed point, we aim to obtain conditions that the function $f(R;\mathcal{L})$ must obey in order to obtain it. The way that the parameters in Eq. (2.15) are defined force us to exclude GR as a possible function, since it would cause them to diverge or be ill-defined.

We can take advantage of already having the value of y and instantly calculate some of the relevant cosmological quantities, since the scale factor $a(t) = e^{H_0 t}$, the density $\rho(t) = \rho_0 e^{-3(1+w)H_0 t}$.

Via the definition of the Ricci scalar (1.11), $R = const. \rightarrow \dot{R} = 0$, which implies that $x = 0$, provided that $f^R \neq 0$, and the first equation of the system (2.16) implies the constraint

$$\theta = -\phi \left[1 + 3(1+w)\left(\beta_L + 1\right)\right]. \tag{2.27}$$

Since the equation for dy/dN simplifies trivially, we are left with the reduced system

$$\begin{cases} \frac{d\phi}{dN} = \phi \left[\phi - 3(1+w)\left(\beta_L + 1\right)\right] \\ \frac{d\theta}{dN} = \theta \left[\phi - 3(1+w)(\gamma_L + 1)\right] \end{cases}, \tag{2.28}$$

subject to the constraint (2.27), assuming that $\alpha_R \neq 0$ and β_R does not diverge.

While applying the above constraint would reduce the dimensionality of the system even further, it would not greatly advance the analysis, since the parameters β_L and γ_L are undetermined until one chooses a specific $f(R,\mathcal{L})$ function.

2.4.1 Empty Universe Solution

We first consider the case where $\rho = 0$, *i.e.* an empty universe. Since we are only studying the fixed points, it does not really matter if this condition is initial or reached asymptotically. In either case, this condition implies that at the fixed point we must have $\theta = \phi = 0$, and both Eq. (2.27) and the system (2.28) are satisfied trivially, as long as β_L and γ_L do not diverge.

This shows that a de Sitter phase with vanishing energy density is always attainable as long as the function $f(R,\mathcal{L})$ does not lead to vanishing parameters α_R, β_R, β_L and γ_L, evaluated at the fixed point $y = 2$, $x = \phi = \theta = 0$.

2.4.2 Non-empty Universe Solution

An interesting characteristic of $f(R,\mathcal{L})$ theories is that it is able to mimic well-known cosmological solutions of GR with very different matter contents. Particularly, it can lead to a matter filled de Sitter universe (note that this was already the case for NMC theories [22–24]).

By denoting the ensuing fixed point(s) as $(x, y, z, \phi, \theta) = (0, 2, z^*, \phi^*, \theta^*)$, and using the simplified dynamical system (2.28) along with the obtained constraint

(2.27), we find that they must obey

$$\begin{cases} z^* = -(1+\phi^*), \\ \theta^* = z^*\phi^* \\ \phi^* = 3(1+w)\left[\beta_L(z^*,\phi^*,\theta^*)+1\right] \\ \beta_L(z^*,\phi^*,\theta^*) = \gamma_L(z^*,\phi^*,\theta^*) \neq 0 \end{cases} . \tag{2.29}$$

More specifically, the final condition translates into the condition $f^{LL}f^{RL} = f^L f^{RLL}$ via the parameter definitions (2.15).

This analysis is hampered by leaving the action undefined, which makes it impossible to calculate the Jacobian matrix of the dynamical system, and hence obtain the stability of this fixed point for a generic $f(R,\mathcal{L})$ function. However, the obtained conditions for the latter should not be overlooked, as they present a quick method of determining if a given models admits a de Sitter solution.

Chapter 3

Dynamical Analysis of Specific $f(R,\mathcal{L})$ Functions

Having established the method to be used to analyse $f(R,\mathcal{L})$ models and derived the dynamical system, we are only left with specifying a function, solving the equations (2.16) for the fixed points, and then calculate the eigenvalues of the Jacobian for each one to check for their stability. The work done in this regard is shown for an exponential function in Section 3.1 and for a power law function in Section 3.2.

3.1 Exponential $f(R,\mathcal{L})$

We now proceed to study the more complex model of an exponential function as given in Ref. [39],

$$f(R,\mathcal{L}) = M^4 \exp\left(\frac{R}{6H_0^2} + \frac{\mathcal{L}}{6H_0^2\kappa}\right), \tag{3.1}$$

where M is a mass scale and H_0 will turn out to be the expansion rate of the fixed point associated with a de Sitter solution. Intuitively, one would assume that this constitutes an extension of GR with a cosmological constant for small R and $\mathcal{L}$. However, a Taylor expansion of the function at $R = \mathcal{L} = 0$ produces a cosmological constant term with the wrong sign, so this analogy is not valid. Switching the signs of M^4, R and $\mathcal{L}$ would produce the correct sign for the cosmological constant, but would also produce inconsistent results later on.

With this $f(R,\mathcal{L})$ function, the dimensionless parameters (2.15) become

$$\begin{aligned} \alpha_R = \beta_R = \gamma_R = \eta_R &= \frac{R}{6H_0^2} = -\frac{y}{z}, \\ \beta_L = \gamma_L &= -\frac{\rho}{6\kappa H_0^2} = \frac{\phi}{3(1+w)}, \end{aligned} \tag{3.2}$$

while the constraints (2.17) yield

$$R = -6H_0^2\frac{y}{z} \quad , \quad \rho = -\frac{6\kappa H_0^2}{3(1+w)}\frac{\theta}{z} \quad , \quad \theta = \phi z. \tag{3.3}$$

The fixed points of the dynamical system (2.16) are shown in Table 3.1, along with the corresponding solutions.

TABLE 3.1: Fixed points and respective solutions for an exponential $f(R,\mathcal{L})$.

Point	(x,y,z,ϕ,θ)	$a(t)$	$\rho(t)$	q
$\mathcal{A}$	$(1,0,0,0,0)$	$\left(\frac{t}{t_0}\right)^{\frac{1}{2}}$	$\rho_0\left(\frac{t}{t_0}\right)^{-\frac{3}{2}(1+w)}$	-1
$\mathcal{B}$	$(0,2,-1,0,0)$	$e^{H_0 t}$	$\rho_0 e^{-3(1+w)H_0 t}$	-1

Point $\mathcal{A}$

Note that both Eq. (1.11) and the relation $R \sim y/z$ imply that the scalar curvature vanishes for $y=0$. However, from the definition (2.9), $x \neq 0$ implies that either $f^R = 0$ or $H = 0$. The former implies

$$f^R = \frac{M^4}{6H_0^2}\left(\frac{R}{6H_0^2} - \frac{\rho}{6\kappa H_0^2}\right) = \frac{M^4}{6H_0^2}\exp\left(-\frac{\rho}{6\kappa H_0^2}\right) = 0, \tag{3.4}$$

which leads to the unphysical result $\rho \to \infty$.

On the other hand, the vanishing scalar curvature implies that $H(t) = 1/2t$. We can only conclude that this point is only valid at $t \to \infty$, where it behaves as a saddle point with no physical interest.

Point $\mathcal{B}$

This solution bears some resemblance to the de Sitter case in GR, both in the form of the scale factor and vanishing energy density for late times, as well as the location of the fixed points. However, the non-minimal coupling introduced by the exponential function make this point unstable, as opposed to the GR case. The expansion rate can be calculated for the definition of the z variable (2.9)

$$z = -\frac{H_0}{H} = -1 \to H = H_0. \tag{3.5}$$

3.2 Power Law $f(R,\mathcal{L})$

We now consider a power law model

$$f(R,\mathcal{L}) = \left(\kappa M^2\right)^{-\varepsilon}(\kappa R + \mathcal{L})^{(1+\varepsilon)}, \tag{3.6}$$

where M is a characteristic mass scale. Contrarily to the exponential model, this one proves to be an extension of GR for small ε.

TABLE 3.2: x, y and z values of the fixed points for a power law $f(R,\mathcal{L})$.

Point	(x,y,z)
$\mathcal{A}$	$(1,0,0)$
$\mathcal{B}$	$(x,0,1-x)$
$\mathcal{C}$	$\left(\frac{3}{2\varepsilon+1}-1, \frac{3}{2\varepsilon+1}-\frac{1}{\varepsilon}+2, \frac{1}{\varepsilon}-\frac{6}{2\varepsilon+1}\right)$
$\mathcal{D}$	$\left(-\frac{3\varepsilon(w+1)(3w-1)}{(\varepsilon+1)[6\varepsilon(w+1)-3w-1]}, \frac{(1-3w)}{2}, \frac{-6\varepsilon(w+1)+3w+1}{2}\right)$

TABLE 3.3: ϕ and θ values of the fixed points for a power law $f(R,\mathcal{L})$.

Point	(ϕ,θ)
$\mathcal{A},\mathcal{B},\mathcal{C}$	$(0,0)$
$\mathcal{D}$	$\left(\frac{3\varepsilon(w+1)[\varepsilon[6\varepsilon(w+1)+3w+5]-2]}{(\varepsilon+1)[6\varepsilon(w+1)-3w-1]}, -\frac{3(w+1)[\varepsilon[6\varepsilon(w+1)+3w+5]-2]}{2}\right)$

In this model, the quantities defined in Eq. (2.15) become

$$\begin{aligned}\alpha_R=\gamma_R=\frac{\varepsilon\eta_R}{1+\varepsilon}=-\frac{\varepsilon\beta_R}{1-\varepsilon} &= \varepsilon\left(1+\frac{\theta}{3(1+w)y\phi-\theta}\right),\\ \beta_L=-\frac{1-\varepsilon}{\varepsilon}\gamma_L &= \frac{(1-\varepsilon)\theta}{3(1+w)y\phi-\theta},\end{aligned}\tag{3.7}$$

while the constraints (2.17) read

$$\begin{aligned}\rho &= \frac{\kappa R}{3(1+w)}\frac{\theta}{y\phi},\\ \phi=\frac{\varepsilon}{1+\varepsilon}\frac{\theta}{z} \quad &, \quad \theta=3(1+w)[(1+\varepsilon)z+y].\end{aligned}\tag{3.8}$$

The results and corresponding physical solutions (obtained via the same process as before) are depicted in Tables 3.2, 3.3 and 3.4.

TABLE 3.4: Cosmological solutions for the fixed points of a power law $f(R,\mathcal{L})$.

Point	$a(t)$	$\rho(t)$	q
$\mathcal{A},\mathcal{B}$	$\left(\frac{t}{t_0}\right)^{\frac{1}{2}}$	$\rho_0\left(\frac{t}{t_0}\right)^{-\frac{3}{2}(1+w)}$	1
$\mathcal{C}$	$\left(\frac{t}{t_0}\right)^{\frac{\varepsilon(1+2\varepsilon)}{1-\varepsilon}}$	$\rho_0\left(\frac{t}{t_0}\right)^{-\frac{3\varepsilon(1+2\varepsilon)(1+w)}{1-\varepsilon}}$	$-1+\frac{1}{\varepsilon}-\frac{3}{1+2\varepsilon}$
$\mathcal{D}$	$\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}$	$\rho_0\left(\frac{t}{t_0}\right)^{-2}$	$\frac{1+3w}{2}$

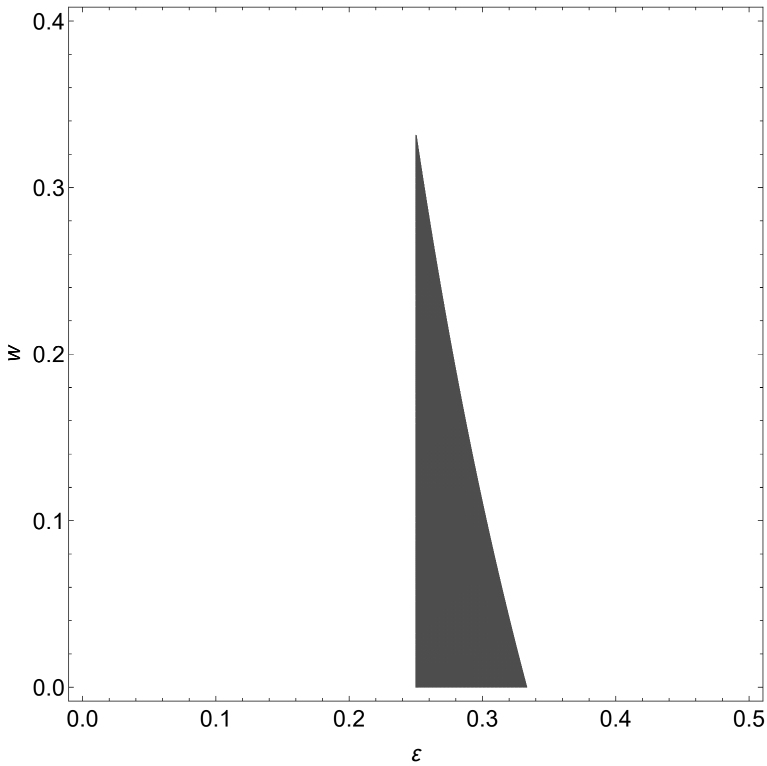

FIGURE 3.1: The dark grey region corresponds to the unstable region of point $\mathcal{A}$. There is no stable region and the remaining phase space corresponds to a saddle point.

Point $\mathcal{A}$

Once again we obtain a point where $x \neq 0$ and $y = 0$, which via the definition of the x variable in Eq. (2.9) implies that either

$$f^R = \kappa(1+\varepsilon)(\kappa M^2)^{-\varepsilon}(R-\rho)^{\varepsilon} = 0, \tag{3.9}$$

or

$$H = 0. \tag{3.10}$$

Both conditions ultimately imply that this solution is only valid at $t \to \infty$, since the scale factor has the same solution as before, so R vanishes and the energy density is inversely proportional to time. The stability of the point is shown in Fig. 3.1. This point presents a decelerated expansion for any type of matter (any value of the EOS parameter w), and since it is either unstable or a saddle point it is not of physical interest as a candidate for dark matter.

Point $\mathcal{B}$

This point has the peculiarity of representing an infinite number of fixed points. It occurs if the exponent ε is related to the EOS parameter w by $\varepsilon = 1/3(1+w)$, which causes the differential equation for x to decouple from the remainder of the system and to become an identity relation, and as such admits any solution.

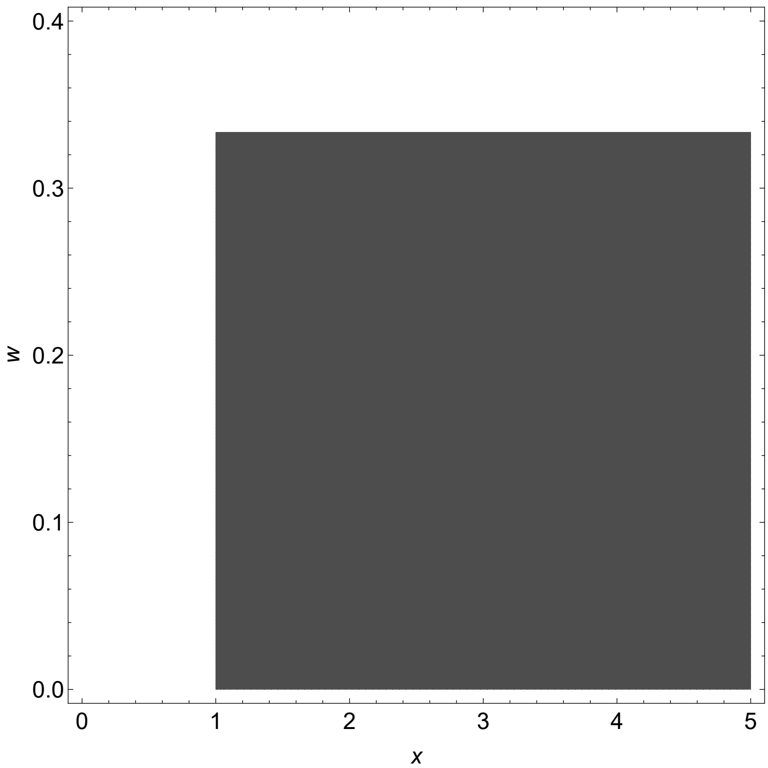

FIGURE 3.2: The dark grey region corresponds to the unstable region of point $\mathcal{B}$. There is no stable region and the remaining phase space corresponds to a saddle point.

However, this point is once more not a viable candidate for dark energy, since it is either unstable or a saddle point, depending on the values of w and x.

Point $\mathcal{C}$

This fixed point proves to be interesting due to having a deceleration parameter that depends on the exponent ε

$$q = -1 + \frac{1}{\varepsilon} - \frac{3}{1+2\varepsilon}, \tag{3.11}$$

depicted in Fig. 3.3. The stability of the point is shown for a range of w and ε values in Fig. 3.4.

Though this point features several stable regions, the only one of interest corresponds to $-1/2 < \varepsilon < 0$. The unstable region does not correspond to any known period in the evolution of the Universe, and the remaining stable regions have $|\varepsilon| > 1$, and thus are not a perturbation to GR, as mentioned previously.

Due to the relation between q and ε near $\varepsilon = 0$, we can have a theory that is arbitrarily close to GR with a negative deceleration parameter driven by regular matter. An odd consequence of this is that the deceleration parameter turns out to be orders of magnitude larger than what one would expect, for example for a de Sitter expansion in GR, which seems to imply that the universe is heading for a

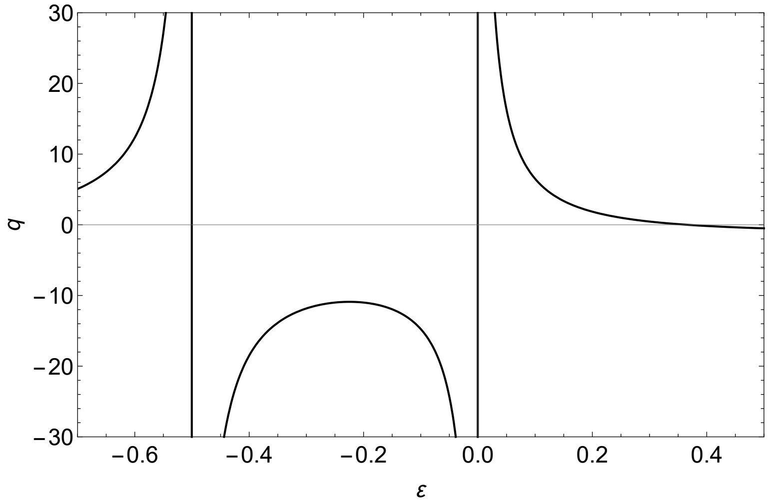

FIGURE 3.3: Deceleration parameter for point $\mathcal{C}$ as a function of the exponent ε.

"big rip". Nonetheless, this hypothesis is not experimentally excluded, so the fixed point remains a viable candidate as a replacement for dark energy.

Point $\mathcal{D}$

Even though this point depends on the value of ε and w, is presents no interest as a candidate for dark energy, as it is not an attractor for any of the phase space. On the other hand, it could have some bearing on inflation, due to it having a deceleration parameter related to the EOS parameter by $q = (1 + 3w)/2$, and thus admits negative values in the unstable region.

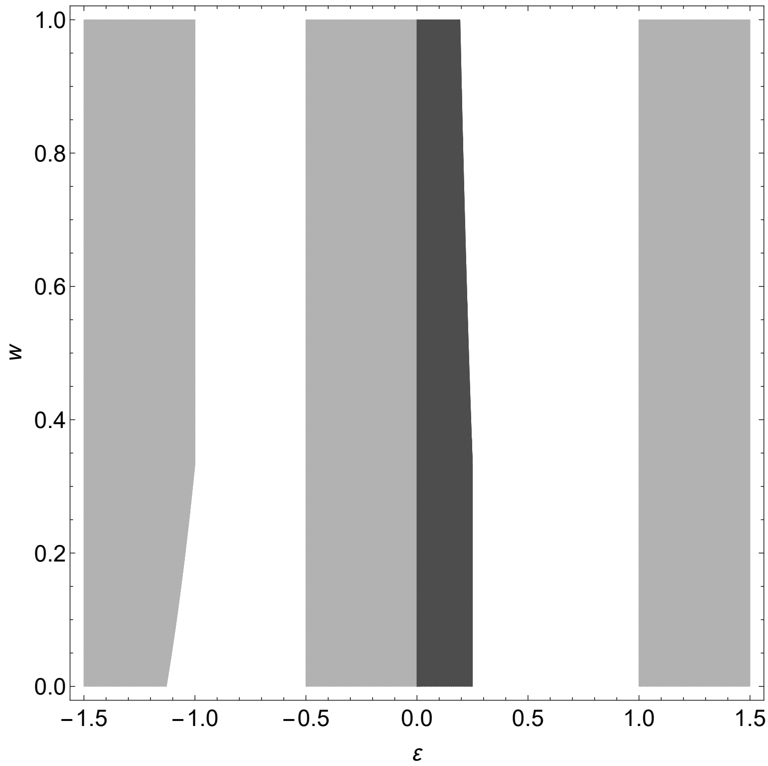

FIGURE 3.4: Stability regions of point $\mathcal{C}$. The dark grey area corresponds to an unstable region, while the light grey corresponds to a stable one. The remaining space corresponds to a saddle point.

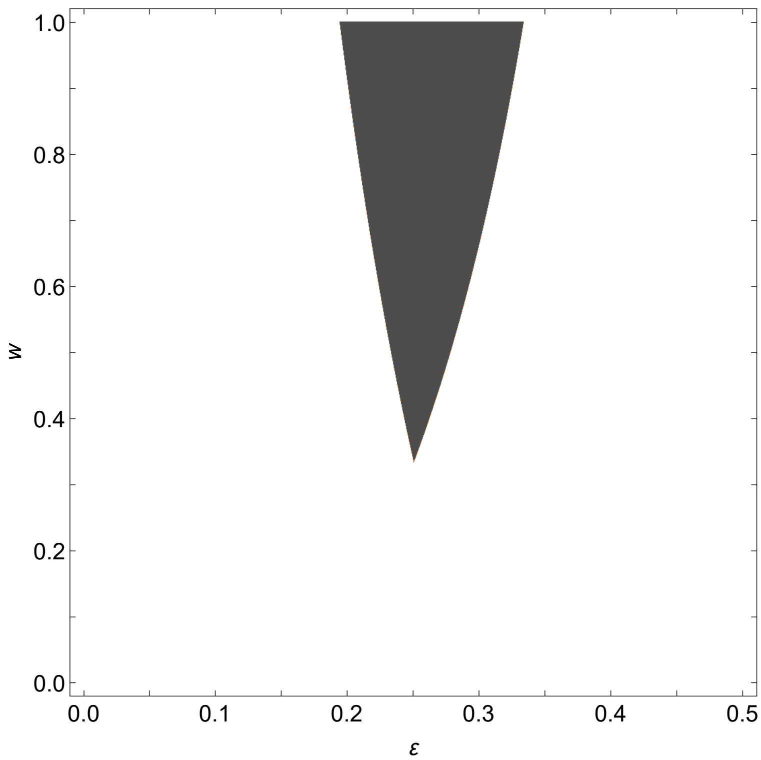

FIGURE 3.5: Stability regions of point $\mathcal{D}$. The dark grey area corresponds to an unstable region and the remaining space corresponds to a saddle point.

Chapter 4

$f(R)(\kappa R+\mathcal{L})$ Theories

We now introduce a new type of theories embodied in the action

$$S=\int d^4x\sqrt{-g}f(R)(\kappa R+\mathcal{L}), \tag{4.1}$$

that is, with $f(R,\mathcal{L})=f(R)(\kappa R+\mathcal{L})$, or equivalently in the NMC formalism of Sec. 1.3, $f_1(R)=f(R)R$ and $f_2(R)=f(R)$. Since the action is a specific case of the latter, we will frequently refer to the results of that Section.

A null variation of the action (4.1) with respect to the metric gives us the filed equations

$$FG_{\mu\nu}=\frac{1}{2}fT_{\mu\nu}+\Delta_{\mu\nu}F+\tfrac{1}{2}g_{\mu\nu}\kappa f-\frac{1}{2}g_{\mu\nu}RF, \tag{4.2}$$

where $F=\kappa f(R)+f'(R)(\kappa R+\mathcal{L})$ and primes denote differentiation with respect to the scalar curvature as usual. Following from Eq. (1.72), the Bianchi identities imply the noncovariant conservation law

$$\nabla^\mu T_{\mu\nu}=\frac{f'}{f}\left(g_{\mu\nu}\mathcal{L}-T_{\mu\nu}\right)\nabla^\mu R. \tag{4.3}$$

4.1 $f(R)(\kappa R+\mathcal{L})$ Cosmology

Once again introducing the cosmological FLRW metric (1.5) and energy-momentum tensor (1.6), one obtains the continuity equation $\nabla^\mu T_{\mu\nu}=0$ and subsequently the conservation of energy. The discussion on energy conservation under a conformal transformation presented in Sec.1.3 and Ref. [45] remains valid.

Separating the 00 and ii components on the field equations (4.2) one respectively obtains the modified Friedmann and Raychaudhuri equations

$$H^2=\frac{1}{3F}\left[\frac{1}{2}FR-3HF'\dot{R}-\tfrac{1}{2}\kappa fR+\frac{1}{2}f\rho-9H^2(1+w)f'\rho\right], \tag{4.4}$$

$$2\dot{H}+3H^2=\frac{1}{2F}\left[FR-\kappa fR-2\ddot{F}-4H\dot{F}-fw\rho\right], \tag{4.5}$$

with $F'\equiv 2\kappa f'+f''(\kappa R-\rho)$.

An interesting exercise is to determine under which conditions these equations result in a de Sitter universe, *i.e.* a scale factor $a(t)=e^{H_0t}$ where H_0 is the present day Hubble parameter. Eq. (1.11) leads to a constant Ricci scalar $R_0 = 12H_0^2$, and the modified Friedmann (4.4) and Raychaudhuri (4.5) equations then posit two scenarios, depending on the value of the energy density ρ.

From Eqs (4.4) and (4.5), one gathers that, if the universe is devoid of any kind of matter, *i.e.* $\rho = 0$, then one must have $f(R_0) = f'(R_0)R_0$, which for an exponential function implies,

$$f(R_0) = \exp\left(\frac{R_0}{M^2}\right) \to R_0 = M^2 \to H_0 = \frac{M}{2\sqrt{3}}. \tag{4.6}$$

On the other hand, if $\rho \neq 0$ one must have

$$f(R_0) = f'(R_0) = 0. \tag{4.7}$$

4.2 Equivalence with Scalar Field Theories

It is not surprising that these theories are equivalent to JBD and scalar field theories similarly to $f(R)$ theories.

The following could be verified by directly substituting on the Eqs. from subsection 1.3.2: however, for convenience, we repeat the procedure. Thus, one can write the action (4.1) as

$$S = \int d^4x\sqrt{-g}\left[f(\chi)(\kappa\chi + \mathcal{L}) + \varphi(R - \chi)\right]. \tag{4.8}$$

Variation with respect to the fields χ and φ return

$$\begin{aligned} &\varphi = \kappa f(\chi) + f'(\chi)(\kappa\chi + \mathcal{L}), \\ &\chi = R, \end{aligned} \tag{4.9}$$

which implies the fields are independent if $f'(\chi)\mathcal{L} \neq 0$. The action can then be written as a JBD theory with $\omega_{JBD} = 0$

$$S = \int d^4x\sqrt{-g}\left[\varphi R - V(\chi, \varphi) + f(\chi)\mathcal{L}\right], \tag{4.10}$$

with a potential

$$V(\chi, \varphi) = \varphi\chi - \kappa\chi f(\chi). \tag{4.11}$$

A more interesting way of rewriting the theory resorts to the conformal transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \varphi g_{\mu\nu}$, so that the action (4.10) can be written in the Einstein frame as

$$\begin{aligned} S = \int d^4x\sqrt{-\tilde{g}}\Big[&\kappa\tilde{R} - \tfrac{1}{2}\partial^\alpha\phi\partial_\alpha\phi - U(\phi,\psi) \\ &+ \psi e^{-2\frac{\phi}{\sqrt{3\kappa}}}\mathcal{L}\left(e^{-\frac{\phi}{\sqrt{3\kappa}}}\tilde{g}_{\mu\nu}, \Psi_M\right)\Big], \end{aligned} \tag{4.12}$$

where the potential is now

$$U(\phi,\psi)=\chi(\psi)\frac{\varphi(\phi)-\kappa\psi}{\varphi(\phi)^2}, \tag{4.13}$$

and we have rescaled the fields as

$$\phi \equiv \sqrt{3\kappa}\ln(\varphi), \tag{4.14}$$

$$\psi \equiv f(\chi). \tag{4.15}$$

4.3 Dynamical System Analysis

Note that the field equations (4.4) and (4.5) can be written in the same way as for the general NMC case (Eqs. (1.77) and (1.78)), with the small distinction of the the dimensionless variables now being defined as

$$\begin{aligned} x=-\frac{F'\dot{R}}{FH} \quad &, \quad y=\frac{R}{6H^2} \quad , \quad z=-\frac{\kappa f R}{6FH^2}, \\ \Omega_1=\frac{f\rho}{6FH^2} \quad &, \quad \Omega_2=-\frac{3(1+w)f'\rho}{F}. \end{aligned} \tag{4.16}$$

As such, they must obey the same autonomous system presented in Sec. 1.3 (presented once more here for convenience)

$$\begin{cases} \frac{dx}{dN} = & x\left[x-y+3(1+w)\left(1+\frac{z}{y}\alpha_1\right)\left(1+\frac{\alpha_2}{\alpha}\right)-3w\right]+2(2+3w)(2-y) \\ & -3(1+w)z(1+\alpha_1)+9(1+w)\frac{z}{y}\alpha_1 \\ \frac{dy}{dN} = & y\left[2(2-y)-\frac{x}{\alpha}\right] \\ \frac{dz}{dN} = & z\left[x\left(1-\frac{\alpha_1}{\alpha}\right)+3(1+w)\left(1+\frac{z}{y}\alpha_1\right)+2(2-y)\right] \end{cases}, \tag{4.17}$$

where the two constraints from Eqs. (1.77) and (1.82) have already been applied and the dimensionless parameters take the form

$$\alpha(R,\rho)=\frac{F'R}{F}, \quad \alpha_1(R)=\frac{f'R}{f}+1, \quad \alpha_2(R)=\frac{f''R}{f'}. \tag{4.18}$$

As before, solving the system generally requires writing finding relations between the relevant physical quantities and the dimensionless variables (4.16). In this specific case, one finds the scalar curvature by inverting

$$\frac{f'(R)R}{f(R)}=-\frac{\Omega_2 y}{3(1+w)\Omega_1}=-\frac{y+z\alpha_1(R)}{\Omega_1}, \tag{4.19}$$

and the energy density from

$$\rho(y,z,\Omega_1)=-\kappa R(y,z,\Omega_1)\frac{\Omega_1}{z}. \tag{4.20}$$

TABLE 4.1: Fixed points for an exponential $f(R)$.

Point	$(x, y, z, \Omega_1, \Omega_2)$
$\mathcal{A}$	$(0, 2, -1, 0, 0)$
$\mathcal{B}$	$(0.552524, 1.60518, -1.24388, 0.327406, -0.241229)$
$\mathcal{C}$	$(0.601438, 1.26626, -1.14062, 0.557947, -0.285022)$

TABLE 4.2: Values of the quantities $r = R/M^2$ and $\varrho = \rho/(kM^2)$ and solutions of the fixed points for an exponential $f(R)$.

Point	(r, ϱ)	$a(t)$	$\rho(t)$	q
$\mathcal{A}$	$(1, 0)$	$e^{H_0 t}$	$\rho_0 e^{-3H_0 t}$	-1
$\mathcal{B}$	$(0.394225, 0.103765)$	$\left(\frac{t}{t_0}\right)^{2.5328}$	$\rho_0 \left(\frac{t}{t_0}\right)^{-7.5984}$	-0.60518
$\mathcal{C}$	$(0.215619, 0.105472)$	$\left(\frac{t}{t_0}\right)^{1.36288}$	$\rho_0 \left(\frac{t}{t_0}\right)^{-4.08864}$	-0.26626

Similarly to the $f(R, \mathcal{L})$ case, in the following section we present the results for two choices of functions $f(R)$: an exponential function, and a power law.

4.4 Exponential $f(R)$

We now proceed to study a model with

$$f(R) = \exp\left(\frac{R}{M^2}\right), \tag{4.21}$$

where M is a mass scale. It is fairly easy to see that this theory collapses to GR for large M or small R. The exponential form of the theory makes it very straightforward to calculate the dimensionless parameters (4.18)

$$\alpha = \frac{2R}{\kappa R - \rho} + \frac{R}{RM^2} \quad , \quad \alpha_1 = 1 + \frac{R}{M^2} \quad , \quad \alpha_2 = \frac{R}{M^2}, \tag{4.22}$$

Due to the complexity of the fixed point solutions, we constrained the results to only include pressure-less matter, *i.e.* $w = 0$. The fixed points associated with this function can be found in Tables 4.1 and 4.2, where $r = R/M^2$ and $\varrho = \rho/(kM^2)$. It should be noted that the values of these dimensionless quantities do not depend on the mass scale M^2.

Point $\mathcal{A}$

The first point corresponds to a stable de Sitter universe with vanishing energy density, whose expansion rate can be calculated from the definition of the z variable

$$z = -\frac{\kappa f R}{6FH^2} = -1 \to H_0 = \frac{M}{2\sqrt{3}}. \tag{4.23}$$

It should be noted that this point is always attained, and stable, for $w \geq 0$, so that an exponential form of the coupling is capable of generating such solutions, regardless of the matter content of the universe.

This is an interesting candidate for dark energy, since it approximates GR for low curvature spacetime, and yet it is able to explain the accelerated expansion of the Universe at late times while resorting only to ordinary matter.

Points $\mathcal{B}$ and $\mathcal{C}$

Both points are stable and present negative deceleration parameters, and as such are both good candidates for dark energy. They differ from point $\mathcal{A}$ in that they do not require a vanishing energy density at the fixed point.

4.5 Power Law $f(R)$

We now consider a power law model

$$f(R) = \left(\frac{R}{M^2}\right)^n, \tag{4.24}$$

where M is mass scale, and that approaches GR if M is large or n is very small. In this case the parameters (4.18) take the form

$$\alpha = \frac{n\left[\kappa(n+1)R - (n-1)\rho\right]}{\kappa(1+n)R - n\rho} \quad , \quad \alpha_1 = n+1 \quad , \quad \alpha_2 = n-1, \tag{4.25}$$

The fixed points and solutions associated with this function can be found in Tables 4.3 and 4.4.

Point $\mathcal{A}$

Point $\mathcal{A}$ is a point whose deceleration parameter depends on the type of matter present in the universe in the same way as in GR. It also requires that ϱ, r, n and w be related by

$$\rho = \kappa R \frac{(n(6n(w+1)+3w+5)-2)}{n(6n(w+1)-3w-1)+3w-1} \tag{4.26}$$

Its stability regions can be seen in Fig. 4.1.

TABLE 4.3: Fixed points for a power law $f(R)$.

Point	$(x, y, z, \Omega_1, \Omega_2)$
$\mathcal{A}$	$\left(\frac{3n(w+1)(n(6n(w+1)+3w+5)+3(w-1))}{3w-1}, \frac{1}{2}(1-3w), \frac{1}{2}(n(6n(w+1)-3w-1)+3w-1), \frac{1}{2}(2-n(6n(w+1)+3w+5)), -\frac{3n(w+1)(n(6n(w+1)+3w+5)-2)}{3w-1}\right)$
$\mathcal{B}$	$\left(-1+\frac{3}{2n+1}, 2-\frac{1}{n}+\frac{3}{2n+1}, \frac{1}{n}-\frac{6}{2n+1}, 0, 0\right)$
$\mathcal{C}$	$(0, 2, 0, -1-3(1+w), 3(1+w))$
$\mathcal{D}$	$\left(\frac{6nw}{1-2n}-4, \frac{n(4n+3w-2)}{(n-1)(2n-1)}, 0, \frac{-4n-3w+2}{2n^2-3n+1}, 3(1+w)\right)$

TABLE 4.4: Cosmological solutions for the fixed points of a power law $f(R)$.

Point	$a(t)$	q
$\mathcal{A}$	$\begin{cases} \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}, & w \neq 1 \\ e^{H_0 t}, & w = 1 \end{cases}$	$\frac{1}{2}(1+3w)$
$\mathcal{B}$	$\begin{cases} \left(\frac{t}{t_0}\right)^{\frac{n(2n+1)}{1-n}}, & n \neq 1 \\ e^{H_0 t}, & n = 1 \end{cases}$	$-1+\frac{1}{n}-\frac{3}{2n+1}$
$\mathcal{C}$	$const.$	
$\mathcal{D}$	$const.$	

While this point has several stable regions, they all require $w < -1/3$ in order to have an accelerated expansion of the universe (again, as in GR), and is thus unsuitable as a candidate for dark energy.

Point $\mathcal{B}$

This point corresponds to a universe with a vanishing energy density $\rho \to 0$ and behaves very similarly to point $\mathcal{C}$ in the case of a power-law $f(R, \mathcal{L})$, with a deceleration parameter given by

$$q = -1 + \frac{1}{n} - \frac{3}{2n+1}, \tag{4.27}$$

and depicted in Fig. 4.2. Its stability can be found in Fig. 4.3.

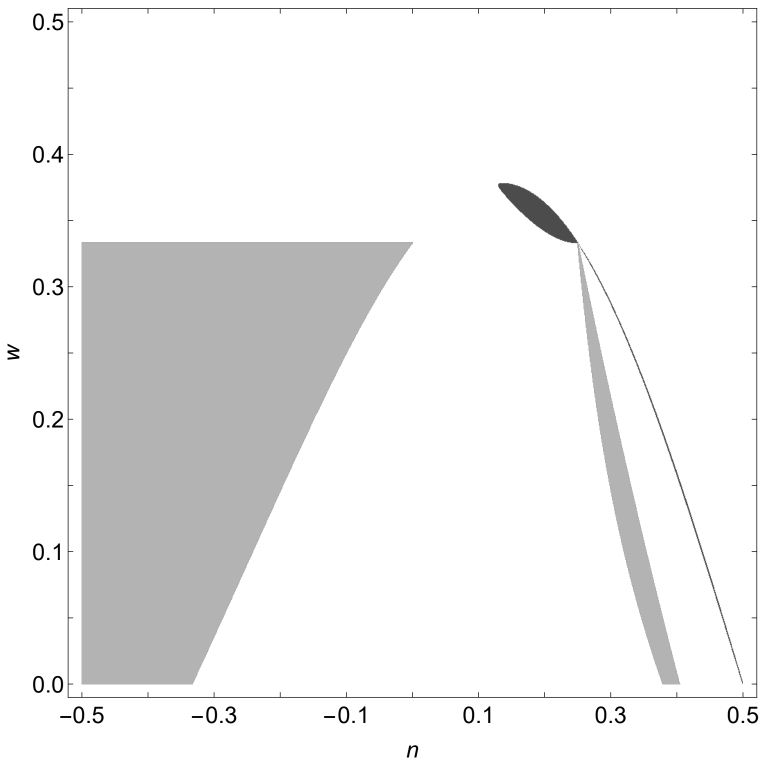

FIGURE 4.1: Stability regions of point $\mathcal{A}$. The dark grey area corresponds to an unstable region, while the light grey corresponds to a stable one. The remaining space corresponds to a saddle point.

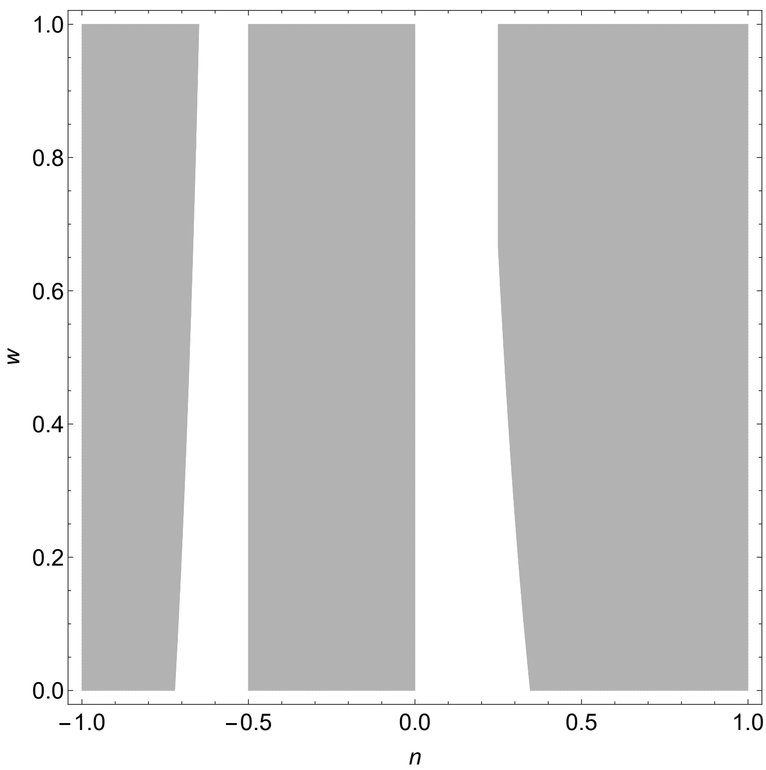

FIGURE 4.3: Stability regions of point $\mathcal{B}$. The light grey corresponds to a stable region and the remaining space corresponds to a saddle point.

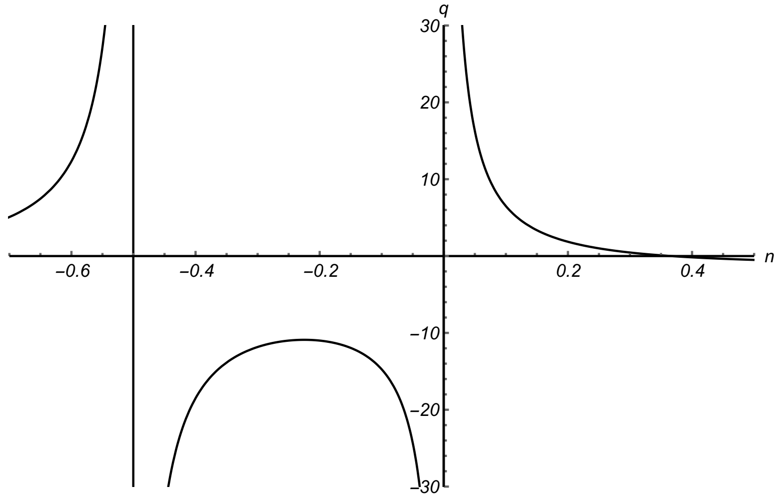

FIGURE 4.2: Deceleration parameter for point $\mathcal{B}$ as a function of the exponent n.

Much like the aforementioned case, this point presents a viable candidate for dark energy, as it can be arbitrarily close to GR while still maintaining a negative deceleration parameter. Furthermore, it also includes the same "big rip" scenario due to one having very large $|q|$ for the stable region with $n \ll 1$. The two stable regions with $|n| > 0.2$ are ignored, due to not having the necessary deceleration parameter.

Point $\mathcal{C}$

This solution is a saddle point with no scalar curvature ($R = 0$) which also requires that n and w be related by $n = 2/(4 + 3w)$. As $R = 0$ and $y = 2$ imply that $H = 0$, this leads to a static universe and a undefined deceleration parameter.

Point $\mathcal{D}$

Similarly to the previous point, point $\mathcal{D}$ has $R = 0$ and $y \neq 0$, implying a static universe and an undefined deceleration parameter. Its stability can be found in Fig. 4.3. It is interesting that this point presents a stable region for an as of yet unobserved evolution of the Universe, which could at first glance suggest that the current accelerated expansion phase is not the final stage in our Universe's evolution.

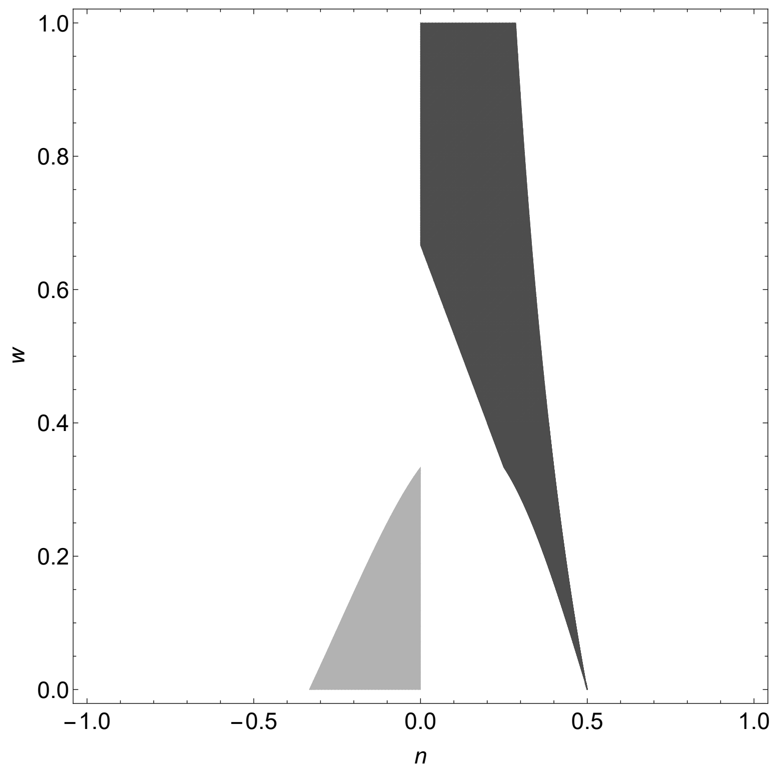

FIGURE 4.4: Stability regions of point $\mathcal{D}$. The dark grey area corresponds to an unstable region, while the light grey corresponds to a stable one. The remaining space corresponds to a saddle point.

Chapter 5

Conclusions

In this thesis the generic case of $f(R,\mathcal{L})$ theories via a dynamical system analysis was studied. It is shown that this model encompasses and is able to replicate the results from GR for $f(R,\mathcal{L}) = \kappa(R-2\Lambda)+\mathcal{L}$, as well as from $f(R)$ and NMC theories, *i.e.* $f(R,\mathcal{L}) = \kappa f(R)+\mathcal{L}$ and $f(R,\mathcal{L}) = \kappa f_1(R)+f_2(R)\mathcal{L}$, respectively.

In Chapter 2 it is shown that a de Sitter expansion can always be obtained for a vanishing energy density, as long the parameters defined in Eq. (2.15) do not diverge. Furthermore, the richness of $f(R,\mathcal{L})$ models is manifest in the possibility of attaining an exponential scale factor driven by a matter-curvature coupling, for specific relations between the derivatives of $f(R,\mathcal{L})$.

As an explanation of the accelerated expansion of the Universe, an exponential $f(R,\mathcal{L})$ fares rather poorly, possessing no attractors with a negative deceleration parameter. Interestingly, it has a fixed point similar to the de Sitter expansion in GR, though this point is made unstable due to the non-minimal coupling induced by the exponential. Furthermore, this theory can not be interpreted as an extension to GR, since a Taylor expansion does not produce the correct signs for the corresponding action.

More interesting is the case of a power-law: while it has two fixed points that have no physical meaning, it also has one that could be related to inflation, and a stable fixed point that can mimic dark energy and be arbitrarily close to GR.

The novel case of a model with Lagrangian density $f(R)\,(\kappa R+\mathcal{L})$ is also presented, and an initial dynamical analysis was performed. While technically a particular case of $f(R,\mathcal{L})$ and NMC theories, the particular coupling presents several interesting properties, such as the form it assumes in the Einstein frame which turns out to be simpler than $f(R)$ theories.

It was found that both exponential and power-law $f(R)$ functions present several fixed points with some explanatory capability for the current evolutionary phase of the Universe, as well as for inflation.

Even though dynamical system analysis proves itself to be extremely useful in cosmology, one must beware several caveats inherent to its formulation. Firstly, the system, and therefore its solutions, is dependent on the choice of variables, so one could omit interesting regimes purely by choosing a specific set of variables in favour of another. Secondly, one must fully understand that the existence of any two fixed points for a given theory does not imply that they are connected by any type of trajectory, as noted in Ref. [42]. Therefore, one cannot assume that any attractor solution which explains the accelerated expansion of the Universe is in fact a global attractor, *i.e.* all trajectories will drive the universe towards that solution, and therefore one may still be subject to a fine-tuning problem.

As a conclusion, this work clearly shows that the models put forward present very interesting possibilities for developing novel cosmological models. Its aim is not to directly account for the observed dynamics of the Universe, but to provide the framework for ulterior studies.

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